Control and Robotics in Medicine
2018-2019

Deliverable D1

September 26, 2018

Deadline: September 25th, 2018 - 8:59
Total mark contribution: 20 %
Modality: Individual

This deliverable is based on the robot of Figure 1.

Figure 1: Laboratory robot.

The degrees of freedom, rotations and reference axes of the robot must be represented as shown in Figure 2. Dimensions of the robot are shown in Table 1 and the mechanical constraints of the rotational angles in Table 2

segment	length (mm)
l0	86.8
l1	31.0
l2	150.2
l3	146.3
l4	70.0
l5	66.3

Table 1: Dimensions of the robot.

rotation	minimum (rad)	maximum (rad)
q1	-2.62	2.62
q2	-0.33	2.97
q3	-2.89	0.26
q4	-1.83	1.86
q5	-1.05	4.19

Table 2: Mechanical constraints of every joint.

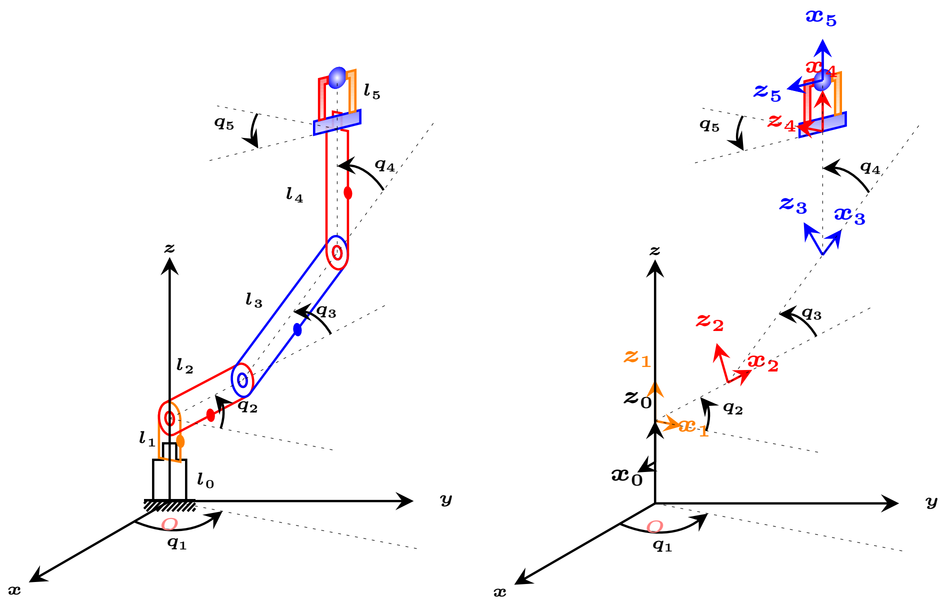

Figure 2: Representation of the degrees of freedom and local coordinate axes of the robot.

Problem definition:

1. **Formulate the forward kinematics problem in position and orientation (25 %).**
   Based on Figure 2, the set of rotational axes of every segment $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$ of the robot with respect to the segment of its kinematic pair is:

$$\mathcal{U} = \{z_0, -y_1, -y_2, -y_3, x_4\} \tag{0.1}$$

Let the set $\widetilde{\mathcal{O}} = \{\tilde{o}_1, \tilde{o}_2, \tilde{o}_3, \tilde{o}_4, \tilde{o}_5\}$ be defined as:

$$\widetilde{\mathcal{O}} = \{(l_0 + l_1)z_0, l_2x_1, l_3x_2, l_4x_3, l_5x_4\} \tag{0.2}$$

We know that (Equation 3.6 of the kinematics document) $\boldsymbol{d_{i-1}^i = R_{i-1}^i \tilde{o}_i}$ and (Equation 3.12 of the kinematics document) $\boldsymbol{d_0^i = d_0^{i-1} + R_0^{i-1} d_{i-1}^i}$. Moreover, (Equation 3.12 of the kinematics document) $\boldsymbol{R_0^i = R_0^{i-1} R_{i-1}^i}$.

Because the formulation of the forward kinematics problem of position consist of obtaining d_0^5, then:

$$d_0^5 = d_0^4 + R_0^4 d_4^5 = d_0^4 + R_0^4 R_4^5 \begin{bmatrix} l_5 \\ 0 \\ 0 \end{bmatrix} \tag{0.3a}$$

$$d_0^4 = d_0^3 + R_0^3 d_3^4 = d_0^3 + R_0^3 R_3^4 \begin{bmatrix} l_4 \\ 0 \\ 0 \end{bmatrix} \tag{0.3b}$$

$$d_0^3 = d_0^2 + R_0^2 d_2^3 = d_0^2 + R_0^2 R_2^3 \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix} \tag{0.3c}$$

$$d_0^2 = d_0^1 + R_0^1 d_1^2 = d_0^1 + R_0^1 R_1^2 \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \tag{0.3d}$$

$$d_0^1 = R_0^1 \begin{bmatrix} 0 \\ 0 \\ (l_0 + l_1) \end{bmatrix} \tag{0.3e}$$

Because of the rotational axes defined by U and the rotational matrices defined in Appendix A.2 of the kinematics document, then:

$$R_0^1 = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.4a}$$

$$R_1^2 = \begin{bmatrix} \cos q_2 & 0 & -\sin q_2 \\ 0 & 1 & 0 \\ \sin q_2 & 0 & \cos q_2 \end{bmatrix} \tag{0.4b}$$

$$R_2^3 = \begin{bmatrix} \cos q_3 & 0 & -\sin q_3 \\ 0 & 1 & 0 \\ \sin q_3 & 0 & \cos q_3 \end{bmatrix} \tag{0.4c}$$

$$R_3^4 = \begin{bmatrix} \cos q_4 & 0 & -\sin q_4 \\ 0 & 1 & 0 \\ \sin q_4 & 0 & \cos q_4 \end{bmatrix} \tag{0.4d}$$

$$R_4^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_5 & -\sin q_5 \\ 0 & \sin q_5 & \cos q_5 \end{bmatrix} \tag{0.4e}$$

If the different rotational matrices with respect to the inertial reference system S_0 are calculated, we obtain:

$$\begin{aligned} R_0^2 = R_0^1 R_1^2 &= \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_2 & 0 & -\sin q_2 \\ 0 & 1 & 0 \\ \sin q_2 & 0 & \cos q_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos q_1 \cos q_2 & -\sin q_1 & -\cos q_1 \sin q_2 \\ \sin q_1 \cos q_2 & \cos q_1 & -\sin q_1 \sin q_2 \\ \sin q_2 & 0 & \cos q_2 \end{bmatrix} \end{aligned} \tag{0.5}$$

$$\begin{aligned} R_0^3 = R_0^2 R_2^3 &= \begin{bmatrix} \cos q_1 \cos q_2 & -\sin q_1 & -\cos q_1 \sin q_2 \\ \sin q_1 \cos q_2 & \cos q_1 & -\sin q_1 \sin q_2 \\ \sin q_2 & 0 & \cos q_2 \end{bmatrix} \begin{bmatrix} \cos q_3 & 0 & -\sin q_3 \\ 0 & 1 & 0 \\ \sin q_3 & 0 & \cos q_3 \end{bmatrix} \\ &= \begin{bmatrix} \cos q_1 (\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & -\sin q_1 & -\cos q_1 (\cos q_2 \sin q_3 + \sin q_2 \cos q_3) \\ \sin q_1 (\cos q_2 \cos q_3 - \sin q_2 \sin q_3) & \cos q_1 & -\sin q_1 (\cos q_2 \sin q_3 + \sin q_2 \cos q_3) \\ \sin q_2 \cos q_3 + \cos q_2 \sin q_3 & 0 & \sin q_2 \sin q_3 + \cos q_2 \cos q_3) \end{bmatrix} \\ &= \begin{bmatrix} \cos q_1 \cos q_{23} & -\sin q_1 & -\cos q_1 \sin q_{23} \\ \sin q_1 \cos q_{23} & \cos q_1 & -\sin q_1 \sin q_{23} \\ \sin q_{23} & 0 & \cos q_{23} \end{bmatrix} \end{aligned} \tag{0.6}$$

where $q_{23} = q_2 + q_3$.

$$\begin{aligned} R_0^4 = R_0^3 R_3^4 &= \begin{bmatrix} \cos q_1 \cos q_{23} & -\sin q_1 & -\cos q_1 \sin q_{23} \\ \sin q_1 \cos q_{23} & \cos q_1 & -\sin q_1 \sin q_{23} \\ \sin q_{23} & 0 & \cos q_{23} \end{bmatrix} \begin{bmatrix} \cos q_4 & 0 & -\sin q_4 \\ 0 & 1 & 0 \\ \sin q_4 & 0 & \cos q_4 \end{bmatrix} \\ &= \begin{bmatrix} \cos q_1 (\cos q_{23} \cos q_4 - \sin q_{23} \sin q_4) & -\sin q_1 & -\cos q_1 (\cos q_{23} \sin q_4 + \sin q_{23} \cos q_4) \\ \sin q_1 (\cos q_{23} \cos q_4 - \sin q_{23} \sin q_4) & \cos q_1 & -\sin q_1 (\cos q_{23} \sin q_4 + \sin q_{23} \cos q_4) \\ \sin q_{23} \cos q_4 + \cos q_{23} \sin q_4 & 0 & \sin q_{23} \sin q_4 + \cos q_{23} \cos q_4) \end{bmatrix} \\ &= \begin{bmatrix} \cos q_1 \cos q_{234} & -\sin q_1 & -\cos q_1 \sin q_{234} \\ \sin q_1 \cos q_{234} & \cos q_1 & -\sin q_1 \sin q_{234} \\ \sin q_{234} & 0 & \cos q_{234} \end{bmatrix} \end{aligned} \tag{0.7}$$

where $q_{234} = q_2 + q_3 + q_4$.

$$
\begin{aligned}
R_0^5 = R_0^4 R_4^5 &= \begin{bmatrix} \cos q_1 \cos q_{234} & -\sin q_1 & -\cos q_1 \sin q_{234} \\ \sin q_1 \cos q_{234} & \cos q_1 & -\sin q_1 \sin q_{234} \\ \sin q_{234} & 0 & \cos q_{234} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_5 & -\sin q_5 \\ 0 & \sin q_5 & \cos q_5 \end{bmatrix} \\
&= \begin{bmatrix} \cos q_1 \cos q_{234} & -\sin q_1 \cos q_5 - \cos q_1 \sin q_{234} \sin q_5 & \sin q_1 \sin q_5 - \cos q_1 \sin q_{234} \cos q_5 \\ \sin q_1 \cos q_{234} & \cos q_1 \cos q_5 - \sin q_1 \sin q_{234} \sin q_5 & -\cos q_1 \sin q_5 - \sin q_1 \sin q_{234} \cos q_5 \\ \sin q_{234} & \cos q_{234} \sin q_5 & cos q_{234} \cos q_5 \end{bmatrix}
\end{aligned} \tag{0.8}
$$

Therefore, if d_0^1 is calculated:

$$
d_0^1 = R_0^1 \begin{bmatrix} 0 \\ 0 \\ (l_0 + l_1) \end{bmatrix} = \begin{bmatrix} \cos q_1 & -sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (l_0 + l_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (l_0 + l_1) \end{bmatrix} = (l_0 + l_1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{0.9}
$$

For d_0^2:

$$
\begin{aligned}
d_0^2 = d_0^1 + R_0^1 d_1^2 = d_0^1 + R_0^1 R_1^2 \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} &= d_0^1 + R_0^2 \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \\
&= d_0^1 + \begin{bmatrix} \cos q_1 \cos q_2 & -\sin q_1 & -\cos q_1 \sin q_2 \\ \sin q_1 \cos q_2 & \cos q_1 & -\sin q_1 \sin q_2 \\ \sin q_2 & 0 & \cos q_2 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \\
&= d_0^1 + l_2 \begin{bmatrix} \cos q_1 \cos q_2 \\ \sin q_1 \cos q_2 \\ \sin q_2 \end{bmatrix}
\end{aligned} \tag{0.10}
$$

For d_0^3:

$$
\begin{aligned}
d_0^3 = d_0^2 + R_0^2 d_2^3 = d_0^2 + R_0^2 R_2^3 \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix} &= d_0^2 + R_0^3 \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix} \\
&= d_0^2 + \begin{bmatrix} \cos q_1 \cos q_{23} & -\sin q_1 & -\cos q_1 \sin q_{23} \\ \sin q_1 \cos q_{23} & \cos q_1 & -\sin q_1 \sin q_{23} \\ \sin q_{23} & 0 & \cos q_{23} \end{bmatrix} \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix} \\
&= d_0^2 + l_3 \begin{bmatrix} \cos q_1 \cos q_{23} \\ \sin q_1 \cos q_{23} \\ \sin q_{23} \end{bmatrix}
\end{aligned} \tag{0.11}
$$

For d_0^4:

$$
\begin{aligned}
d_0^4 = d_0^3 + R_0^3 d_3^4 = d_0^3 + R_0^3 R_3^4 \begin{bmatrix} l_4 \\ 0 \\ 0 \end{bmatrix} &= d_0^3 + R_0^4 \begin{bmatrix} l_4 \\ 0 \\ 0 \end{bmatrix} \\
&= d_0^3 + \begin{bmatrix} \cos q_1 \cos q_{234} & -\sin q_1 & -\cos q_1 \sin q_{234} \\ \sin q_1 \cos q_{234} & \cos q_1 & -\sin q_1 \sin q_{234} \\ \sin q_{234} & 0 & \cos(q_{234} \end{bmatrix} \begin{bmatrix} l_4 \\ 0 \\ 0 \end{bmatrix} \\
&= d_0^3 + l_4 \begin{bmatrix} \cos q_1 \cos q_{234} \\ \sin q_1 \cos q_{234} \\ \sin q_{234} \end{bmatrix}
\end{aligned} \tag{0.12}
$$

For d_0^5:

$$d_0^5 = d_0^4 + R_0^4 d_4^5 = d_0^4 + R_0^4 R_4^5 \begin{bmatrix} l_5 \\ 0 \\ 0 \end{bmatrix} = d_0^4 + R_0^5 \begin{bmatrix} l_5 \\ 0 \\ 0 \end{bmatrix}$$
$$= d_0^4 + \begin{bmatrix} \cos q_1 \cos q_{234} & -\sin q_1 \cos q_5 - \cos q_1 \sin q_{234} \sin q_5 & \sin q_1 \sin q_5 - \cos q_1 \sin q_{234} \cos q_5 \\ \sin q_1 \cos q_{234} & \cos q_1 \cos q_5 - \sin q_1 \sin q_{234} \sin q_5 & -\cos q_1 \sin q_5 - \sin q_1 \sin q_{234} \cos q_5 \\ \sin q_{234} & \cos q_{234} \sin q_5 & \cos q_{234} \cos q_5 \end{bmatrix} \begin{bmatrix} l_5 \\ 0 \\ 0 \end{bmatrix}$$
$$= d_0^4 + l_5 \begin{bmatrix} \cos q_1 \cos q_{234} \\ \sin q_1 \cos q_{234} \\ \sin q_{234} \end{bmatrix} \tag{0.13}$$

Therefore, the forward kinematics problem of position is formulated as

$$d_0^5 = (l_0 + l_1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + l_2 \begin{bmatrix} \cos q_1 \cos q_2 \\ \sin q_1 \cos q_2 \\ \sin q_2 \end{bmatrix} + l_3 \begin{bmatrix} \cos q_1 \cos q_{23} \\ \sin q_1 \cos q_{23} \\ \sin q_{23} \end{bmatrix} + l_4 \begin{bmatrix} \cos q_1 \cos q_{234} \\ \sin q_1 \cos q_{234} \\ \sin q_{234} \end{bmatrix} + l_5 \begin{bmatrix} \cos q_1 \cos q_{234} \\ \sin q_1 \cos q_{234} \\ \sin q_{234} \end{bmatrix}$$
$$= (l_0 + l_1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + l_2 \begin{bmatrix} \cos q_1 \cos q_2 \\ \sin q_1 \cos q_2 \\ \sin q_2 \end{bmatrix} + l_3 \begin{bmatrix} \cos q_1 \cos q_{23} \\ \sin q_1 \cos q_{23} \\ \sin q_{23} \end{bmatrix} + (l_4 + l_5) \begin{bmatrix} \cos q_1 \cos q_{234} \\ \sin q_1 \cos q_{234} \\ \sin q_{234} \end{bmatrix} \tag{0.14}$$

and the orientation as:

$$R_0^5 = \begin{bmatrix} \cos q_1 \cos q_{234} & -\sin q_1 \cos q_5 - \cos q_1 \sin q_{234} \sin q_5 & \sin q_1 \sin q_5 - \cos q_1 \sin q_{234} \cos q_5 \\ \sin q_1 \cos q_{234} & \cos q_1 \cos q_5 - \sin q_1 \sin q_{234} \sin q_5 & -\cos q_1 \sin q_5 - \sin q_1 \sin q_{234} \cos q_5 \\ \sin q_{234} & \cos q_{234} \sin q_5 & cos q_{234} \cos q_5) \end{bmatrix} \tag{0.15}$$

2. **Inverse kinematics problem (75 %).**

   (a) **Formulate the inverse kinematics problem by using the kinematic decoupling technique** (60 %).

   Taking into account the technique of the kinematic decoupling, C can be defined, with respect to the inertial system, as:

$$\overrightarrow{OC} = \overrightarrow{OQ} - (l_4 + l_5)\boldsymbol{a} \tag{0.16}$$

   Because of the forward kinematics obtained in the previous section, being the vector $\overrightarrow{OC}$ with respect to inertial system the d_0^3 vector previously obtained, then:

$$\overrightarrow{OC} = d_0^3 = (l_0 + l_1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + l_2 \begin{bmatrix} \cos q_1 \cos q_2 \\ \sin q_1 \cos q_2 \\ \sin q_2 \end{bmatrix} + l_3 \begin{bmatrix} \cos q_1 \cos q_2 3 \\ \sin q_1 \cos q_2 3 \\ sin q_2 3 \end{bmatrix}$$
$$= \begin{bmatrix} l_2 \cos q_1 \cos q_2 + l_3 \cos q_1 \cos q_{23} \\ l_2 \sin q_1 \cos q_2 + l_3 \sin q_1 \cos q_{23} \\ (l_0 + l_1) + l_2 \sin q_2 + l_3 \sin q_{23} \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \tag{0.17}$$

   Then, it is mandatory to solve the following equations' system:

$$\begin{cases} C_x = l_2 \cos q_1 \cos q_2 + l_3 \cos q_1 \cos q_{23} \\ C_y = l_2 \sin q_1 \cos q_2 + l_3 \sin q_1 \cos q_{23} \\ C_z = (l_0 + l_1) + l_2 \sin q_2 + l_3 \sin q_{23} \end{cases} \tag{0.18}$$

   that can be represented as:

$$\begin{cases} C_x = \cos q_1 (l_2 \cos q_2 + l_3 \cos q_{23}) \\ C_y = \sin q_1 (l_2 \cos q_2 + l_3 \cos q_{23}) \\ C_z = (l_0 + l_1) + l_2 \sin q_2 + l_3 \sin q_{23} \end{cases} \tag{0.19}$$

To obtain q_1, we divide the second and the first equations:

$$\frac{Cy}{C_x} = \frac{\sin q_1}{\cos q_1} = \tan q_1 \rightarrow q_1 = \arctan \frac{C_y}{C_x} \tag{0.20}$$

To obtain q_2 it is mandatory to solve the following equations' system:

$$\begin{cases} \frac{C_x}{\cos q_1} = l_2 \cos q_2 + l_3 \cos q_{23} \\ C_z - (l_0 + l_1) = l_2 \sin q_2 + l_3 \sin q_{23} \end{cases} \tag{0.21}$$

This system can be developed as follows.

$$\begin{cases} \frac{C_x}{\cos q_1} - l_2 \cos q_2 = l_3 \cos q_{23} \\ C_z - (l_0 + l_1) - l_2 \sin q_2 = l_3 \sin q_{23} \end{cases} \tag{0.22}$$

Squaring both terms:

$$\begin{cases} (\frac{C_x}{\cos q_1} - l_2 \cos q_2)^2 = l_3^2 \cos^2 q_{23} \\ (C_z - (l_0 + l_1) - l_2 \sin q_2)^2 = l_3^2 \sin^2 q_{23} \end{cases} \tag{0.23}$$

By adding both equations:

$$l_3^2 = (\frac{C_x}{\cos q_1} - l_2 \cos q_2)^2 + (C_z - (l_0 + l_1) - l_2 \sin q_2)^2 \tag{0.24}$$

If the previous equation is developed:

$$\frac{C_x}{\cos q_1} \cos q_2 + (C_z - (l_0 + l_1)) \sin q_2 = \frac{l_2^2 + (\frac{C_x}{\cos q_1})^2 + (C_z - (l_0 + l_1))^2 - l_3^2}{2l_2} \tag{0.25}$$

By following the same instructions as in Appendix B of the kinematics document and taking into account Equation B.6, it can be obtained that:

$$q_2 = -\beta + \arctan \frac{L}{\pm\sqrt{1 - L^2}} \tag{0.26}$$

where

$$\beta = \arctan \frac{\frac{C_x}{\cos q_1}}{C_z - (l_0 + l_1)} \tag{0.27a}$$

$$L = \frac{l_2^2 + (\frac{C_x}{\cos q_1})^2 + (C_z - (l_0 + l_1))^2 - l_3^2}{2l_2\sqrt{(\frac{C_x}{\cos q_1})^2 + (C_z - (l_0 + l_1))^2}} \tag{0.27b}$$

To obtain q_3, we start from the same equations as in q_2:

$$\begin{cases} \frac{C_x}{\cos q_1} - l_2 \cos q_2 = l_3 \cos q_{23} \\ C_z - (l_0 + l_1) - l_2 \sin q_2 = l_3 \sin q_{23} \end{cases} \tag{0.28}$$

By dividing one by the other,

$$\tan q_{23} = \frac{C_z - (l_0 + l_1) - l_2 \sin q_2}{\frac{C_x}{\cos q_1} - l_2 \cos q_2} \tag{0.29}$$

Therefore,

$$q_3 = \arctan \frac{C_z - (l_0 + l_1) - l_2 \sin q_2}{\frac{C_x}{\cos q_1} - l_2 \cos q_2} - q_2 \tag{0.30}$$

For the calculation of the inverse kinematics of orientation, by taking into account Section 6 of the kinematics document, it is known that:

$$R_{S_A} = R_0^3 = \begin{bmatrix} \cos q_1 \cos q_{23} & -\sin q_1 & -\cos q_1 \sin q_{23} \\ \sin q_1 \cos q_{23} & \cos q_1 & -\sin q_1 \sin q_{23} \\ \sin q_{23} & 0 & \cos q_{23} \end{bmatrix} \tag{0.31}$$

Let matrix R_{S_0} be defined as:

$$R_{S_0} = \begin{bmatrix} a_x & -n_x & -s_x \\ a_y & -n_y & -s_y \\ a_z & -n_y & -s_z \end{bmatrix} \tag{0.32}$$

Being $R_{S_H} = R_{S_A}^T R_{S_0}$, where

$$R_{S_A}^T = \begin{bmatrix} \cos q_1 \cos q_{23} & \sin q_1 \cos q_{23} & \sin q_{23} \\ -\sin q_1 & \cos q_1 & 0 \\ -\cos q_1 \sin q_{23} & -\sin q_1 \sin q_{23} & \cos q_{23} \end{bmatrix} \tag{0.33}$$

and representing R_{S_M} as

$$R_{S_H} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \tag{0.34}$$

where

$$\begin{aligned} u_{11} &= a_x \cos q_1 \cos q_{23} + a_y \sin q_1 \cos q_{23} + a_z \sin q_{23} \\ u_{12} &= -n_x \cos q_1 \cos q_{23} - n_y \sin q_1 \cos q_{23} - n_z \sin q_{23} \\ u_{13} &= -s_x \cos q_1 \cos q_{23} - s_y \sin q_1 \cos q_{23} - s_z \sin q_{23} \\ u_{21} &= -a_x \sin q_1 + a_y \cos q_1 \\ u_{22} &= n_x \sin q_1 - n_y \cos q_1 \\ u_{23} &= s_x \sin q_1 - s_y \cos q_1 \\ u_{31} &= -a_x \cos q_1 \sin q_{23} - a_y \sin q_1 \sin q_{23} + a_z \cos q_{23} \\ u_{32} &= n_x \cos q_1 \sin q_{23} + n_y \sin q_1 \sin q_{23} - n_z \cos q_{23} \\ u_{33} &= s_x \cos q_1 \sin q_{23} + s_y \sin q_1 \sin q_{23} - n_z \cos q_{23} \end{aligned} \tag{0.35}$$

Moreover, it is known that

$$R_{S_H} = R_3^5 = R_3^4 R_4^5 = \begin{bmatrix} \cos q_4 & 0 & -\sin q_4 \\ 0 & 1 & 0 \\ \sin q_4 & 0 & \cos q_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_5 & -\sin q_5 \\ 0 & \sin q_5 & \cos q_5 \end{bmatrix} = \begin{bmatrix} \cos q_4 & -\sin q_4 \sin q_5 & -\sin q_4 \cos q5 \\ 0 & \cos q_5 & -\sin q_5 \\ \sin q_4 & \cos q_4 \sin q_5 & \cos q_4 \cos q_5 \end{bmatrix} \tag{0.36}$$

So, we can define

$$q_4 = \arctan \frac{u_{31}}{u_{11}} \tag{0.37a}$$

$$q_5 = \arctan \frac{-u_{23}}{u_{22}} \tag{0.37b}$$

Therefore,

$$q_4 = \arctan \frac{-a_x \cos q_1 \sin q_{23} - a_y \sin q_1 \sin q_{23} + a_z \cos q_{23}}{a_x \cos q_1 \cos q_{23} + a_y \sin q_1 \cos q_{23} + a_z \sin q_{23}} \tag{0.38a}$$

$$q_5 = \arctan \frac{s_y \cos q_1 - s_x \sin q_1}{n_x \sin q_1 - n_y \cos q_1} \tag{0.38b}$$

In summary, this is the formulation of the inverse kinematics problem:

$$q_1 = \arctan \frac{C_y}{C_x} \tag{0.39a}$$

$$q_2 = -\beta + \arctan \frac{L}{\pm\sqrt{1-L^2}} \tag{0.39b}$$

where $\beta = \arctan \dfrac{\frac{C_x}{\cos q_1}}{C_z - (l_0 + l_1)}$ y $L = \dfrac{l_2^2 + (\frac{C_x}{\cos q_1})^2 + (C_z - (l_0+l_1))^2 - l_3^2}{2l_2\sqrt{(\frac{C_x}{\cos q_1})^2 + (C_z - (l_0+l_1))^2}}$

$$q_3 = \arctan \frac{C_z - (l_0 + l_1) - l_2 \sin q_2}{\frac{C_x}{\cos q_1} - l_2 \cos q_2} - q_2 \tag{0.39c}$$

$$q_4 = \arctan \frac{-a_x \cos q_1 \sin q_{23} - a_y \sin q_1 \sin q_{23} + a_z \cos q_{23}}{a_x \cos q_1 \cos q_{23} + a_y \sin q_1 \cos q_{23} + a_z \sin q_{23}} \tag{0.39d}$$

$$q_5 = \arctan \frac{s_y \cos q_1 - s_x \sin q_1}{n_x \sin q_1 - n_y \cos q_1} \tag{0.39e}$$

(b) **Solve the inverse kinematics problem when** $Q(t_g) = (250, 150, 150)$**,** $a(t_g) = \begin{bmatrix}0.8575 & 0.5145 & 0\end{bmatrix}^T$ **and** $s(t_g) = \begin{bmatrix}-0.5145 & 0.8575 & 0\end{bmatrix}^T$.

$C(t_g) = Q(t_g) - (l_4 + l_5)a = \begin{bmatrix}250 - 0.8575(l_4+l_5) & 150 - 0.5145(l_4+l_5) & 150)\end{bmatrix}^T = \begin{bmatrix}133.12 & 79.87 & 150\end{bmatrix}^T$.

$n = s \text{ x } a = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T$

By applying the equations obtained in the previous section and based on the values defined in Table 1, we obtain:

$$\begin{cases} q_1 = 0.5404 \\ q_2 = 1.1904 \\ q_3 = -2.0135 \\ q_4 = 0.8230 \\ q_5 = \pi/2 \end{cases} \tag{0.40}$$

that belong to the limits defined in Table 2.

(c) **Solve the inverse kinematics problem when** $Q(t_r) = (0, 220, 150)$**,** $a(t_r) = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T$ **and** $s(t_r) = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$. Units are in millimeters (20 %).

$C(t_g) = Q(t_g) - (l_4 + l_5)a = \begin{bmatrix}0 & 220 & 150 - (l_4+l_5)\end{bmatrix}^T = \begin{bmatrix}0 & 220 & 13.7\end{bmatrix}^T$.

$n = s \text{ x } a = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T$

By applying the equations obtained in the previous section and based on the values defined in Table 1, we obtain:

$$\begin{cases} q_1 = \pi/2 \\ q_2 = 1.0122 \\ q_3 = -0.7273 \\ q_4 = -1.8557 \\ q_5 = \pi \end{cases} \tag{0.41}$$

that belong to the limits defined in Table 2.