Kinematics

Félix Monasterio-Huelin & Álvaro Gutiérrez Martín

September 8, 2017

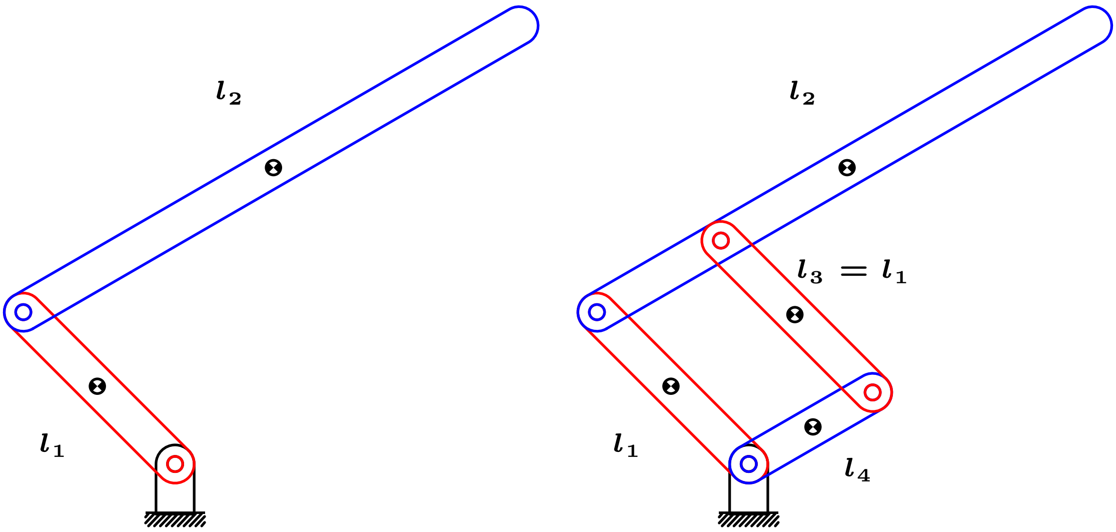
Contents

List of Figures

List of Tables

1 Degrees of freedom and kinematic chains of rigid body mechanisms

The concept of **Degree of Freedom** (DOF) has different meanings. In this document, it will be exclusively used in the sense of the mobility of a mechanism. This mechanism is considered to be composed of a set of rigid bodies connected by any type of joint, that is, creating a **kinematic chain** of rigid bodies.



(a) Two DOFs planar serial mechanism. (b) Five bars Two DOFs planar parallel mechanism.

Figure 1.1: Two DOFs planar mechanisms.

Figure 1.1 shows two planar mechanisms of two DOFs. Figure 1.1a shows a serial mechanism as an **open kinematic chain** while Figure 1.1b shows a parallel mechanism as a **closed kinematic chain**. Serial mechanisms are always composed of open kinematic chains, while in parallel mechanisms there is, at least, one closed kinematic chain. Mechanisms are defined as parallel when the end effector (l_2 in Figure 1.1b) is connected to more than one independent mechanism: l_1 integrates a mechanism and l_3 and l_4 integrate another one. An advantage of parallel mechanisms is that motors can be attached to a common surface, while in serial mechanisms motors must be attached to each joint. As a consequence, parallel mechanisms are typically lighter than serial ones and reduce the inherent flexibility of a rigid bar.

A **kinematic pair** is a connection between two rigid bodies that imposes constraints on their relative movement. A kinematic pair is defined by its DOFs, but it is also typical to talk about the DOFs of the joint of the kinematic par. Therefore, it is the same to talk about a rotational joint of 1 DOF than a kinematic pair with a rotational joint of 1 DOF.

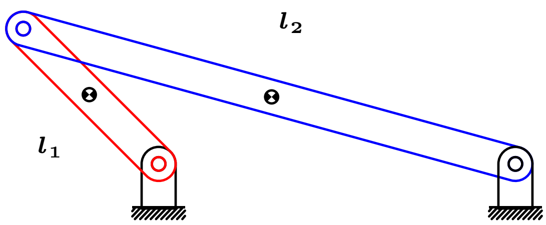


Figure 1.2: Zero DOFs planar mechanism.

The **mobility** of a mechanism is a kinematic concept that indicates the capacity that the mechanism has to perform infinitesimal movements. The **number of DOFs** is the minimum number of independent kinematic variables that allows a mechanism to move. For example, the mechanism of Figure 1.2 consists of a closed planar kinematic chain made up of two bars joined to the ground by 3 rotational joints. It has no mobility, so the number of DOFs equals zero.

The number of DOFs of a rigid body that moves freely in the three-dimensional Cartesian space equals 6: three for the position in $\mathbb{R}^3$ and three for the orientation in $SO(3)$ (creating the $SE(3) =$

$\mathbb{R}^3 \times SO(3)$ space, called group of spatial rigid transformations which dimension is $\mathcal{D} = 6$). If movements are limited in such a way that it remains in the plane, the number of DOFs is reduced to 3: two for the position in $\mathbb{R}^2$ and one for the orientation in $SO(2)$ (creating the $SE(2) = \mathbb{R}^2 \times SO(2)$ space, called group of planar rigid transformations which dimension is $\mathcal{D} = 3$). Independently of how the plane is limited, the system can be defined as three independent equations that only depend on the position and orientation. The rotational space has changed from $SO(3)$, with $\mathcal{D} = 3$, to $SO(2)$, with $\mathcal{D} = 1$ and the Cartesian space has changed from $\mathbb{R}^3$ to $\mathbb{R}^2$. Moreover, if the system is limited to perform rotations around a perpendicular axis to the plane (rotation kinematic pair R) or translation motion following a line contained in the plane (translational or prismatic kinematic pair P), the number of DOFs will be reduced to 1. On the first case, this is because it cannot perform any translational movement, so it can only perform rotations on a subspace of $\mathcal{D} = 1$, that is, a circumference. On the second case, this is because it cannot perform any rotational movement so it can only perform translations on a subspace of $\mathcal{D} = 1$, that is, a line.

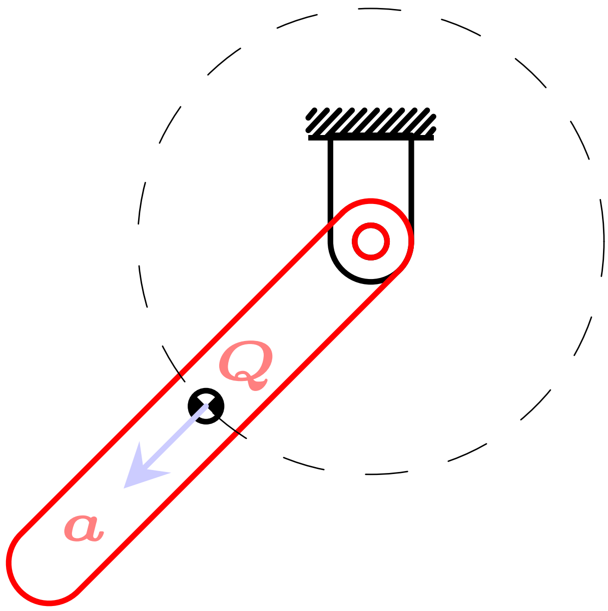


Figure 1.3: One DOF plan mechanism: simple pendulum.

Figure 1.3 shows the first case, a simple pendulum. It is a 1 DOF mechanism where the center of mass (point Q) can only be moved in a circumference of constant radio. To change its orientation ($\vec{a}$ vector), it is mandatory to change the position, so there are two constraints in the Cartesian plane.

In general, it is not easy to know the number of constraints of a three-dimensional mechanism. However, if a mechanical system in $\mathbb{R}^3$ has a set of N rigid bodies and m constraints than only depend on the position $r_i \in \mathbb{R}^3$, $i = 1 \dots N$ and the orientation $R_i \in SO(3)$ defined by integrable and independent vector functions such as $g_k(r, R) = 0$, $k = 1 \dots m$, then the Implicit Function Theorem shows that the system has $n = 6N - m$ DOFs.

$$n = 6N - m \tag{1.1}$$

As an example, a mechanism as the one shown in Figure 1.1a, who has 2 segments (without taking into account the bar in the ground), $N = 2$, and two rotational joints (of 1 DOF each), has 2 DOFs ($n = 2$). This is because the number of constraints (in the three-dimensional space $\mathbb{R}^3$) of each joint is 5, therefore $m = 10$.

In the literature, several studies have focused on the calculation of the number of DOFs of a rigid body attending to the number of elements, number of joints or kinematic pairs and the DOFs of every joint or kinematic pair. The Grübler-Kutzbach mobility criteria is the most known. However it does not calculates correctly the number of DOFs of every mechanism.

The Grübler-Kutzbach equation is written as follows:

$$n = \mathcal{D}(\mathcal{N} - 1 - j) + \sum_{i=1}^{j} f_i \tag{1.2}$$

where $\mathcal{D}$ is the space dimension in which the mechanism is located ($\mathcal{D} = 6$ in a three-dimensional space and $\mathcal{D} = 3$ in the plane), $\mathcal{N}$ is the number of elements in which the anchor to the ground is counted, j is the number of joints and f_i is the number of DOFs of the $i-th$ joint.

Equation 1.2 is valid for many mechanisms (simple open ($j = \mathcal{N} - 1$) and closed ($j = \mathcal{N}$) kinematic chains included). The closed kinematic chain planar mechanism ($\mathcal{D} = 3$) of Figure 1.1b has $n = 2$ DOFs because it has five segments (taking into account the anchor to the ground), $\mathcal{N} = 5$, and five rotations joints, $j = 5$, of 1 DOF each, $f_i = 1$.

Figure 1.4 shows a 3 DOFs serial mechanism as a simple open kinematic chain which satisfies the Grübler-Kutzbach equation: $\mathcal{D} = 6$, $\mathcal{N} = 4$, $j = 3$ and $f_i = 1$ with $i = 1, 2, 3$.

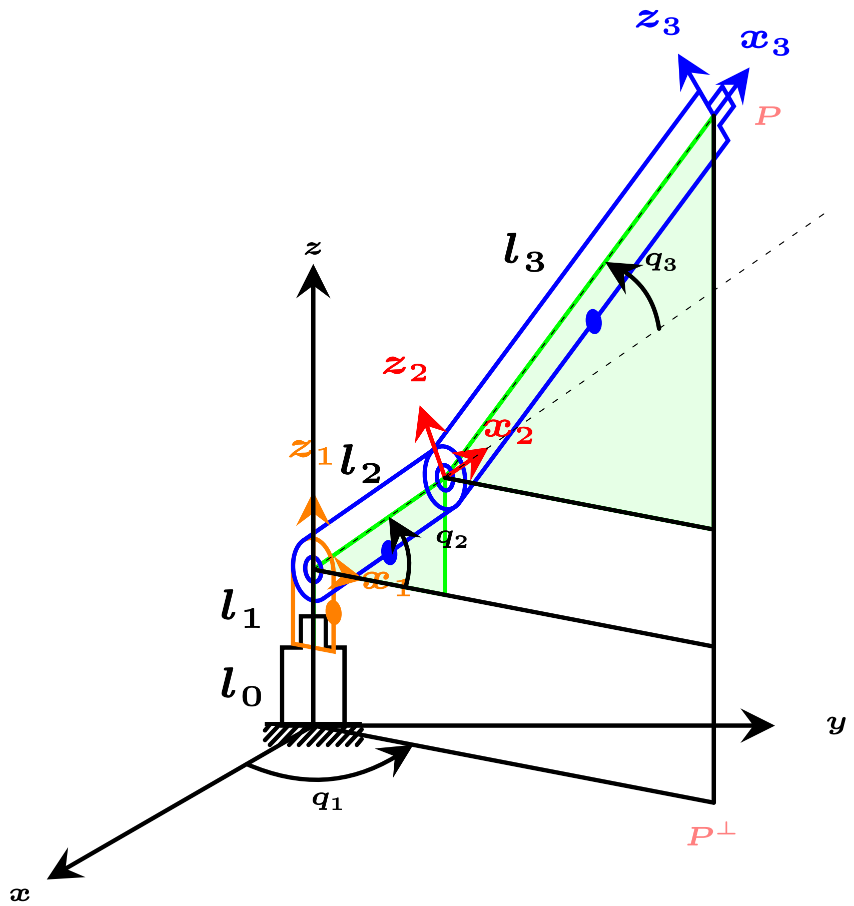


Figure 1.4: 3 DOFs three-dimensional mechanism.

Every DOF is associated to q_i, $i = 1 \dots n$, which will be named in this text as **generalized coordinate** or joint coordinate. These coordinates are defined in the **mechanism configuration space** which dimension is the number of DOFs (n) of the mechanism.

2 Six degrees of freedom robot

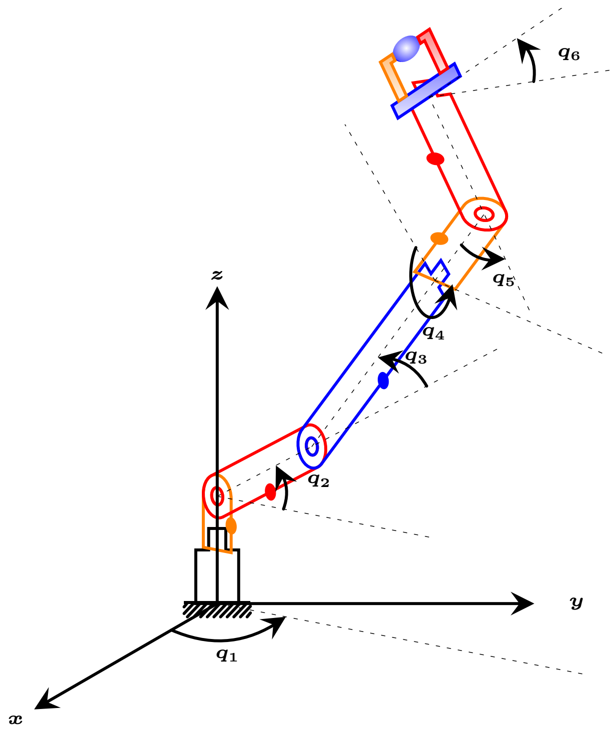


Figure 2.1: 6 DOFs robot.

Figure 2.1 shows the robot to be studied in this document. It has 6 DOFs $\{q_1, q_2, q_3, q_4, q_5, q_6\}$. In Section 8 a specific case is studied, where the movement of q_4 is restricted with the objective of analyzing a 5 DOFs robot.

The specific generalized coordinates' values shown in Figure 2.1 are $\left\{\frac{\pi}{3}, \frac{5\pi}{36}, \frac{\pi}{9}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}\right\}$. Moreover, Figure 2.1 shows which are the rotation axes of each segment relative to the previous one. Section 3 defines more exhaustively the reference system of every segment.

In general, any set of generalized coordinates can be selected as long as they are independent of each other. In fact, every generalized coordinate should be an invertible lineal combination of the ones defined in this document. It is not mandatory that the generalized coordinates are relative angles between adjacent segments. For example, the third generalized coordinate could have been defined as $\theta_3 = q_2 + q_3$, and the set $\{q_1, q_2, \theta_3, q_4, q_5, q_6\}$ will still represent the generalized coordinates of the same robot. In practice, the selection of one system or another depends on the type of robot and the study to be done. The selection of Figure 2.1 is done because the actuators will be located at every joint, so it is the most convenient representation from the control point view.

In the remainder of this document, the first three generalized coordinates $\{q_1, q_2, q_3\}$ will be called the arm or elbow and the last three ones $\{q_4, q_5, q_6\}$ will be called the hand or spherical wrist. Section 5 will show how a robot composed of these two structures simplifies the **inverse kinematics problem**. That is, the problem of obtaining the generalized coordinates when the position and rotation, in the Cartesian space, of the hand's end are known. In any robotic problem related to a task of grasping, transporting, pushing, etc. and provided that there are no exteroceptive sensors (such as vision, touch, etc., that is, sensors relative to the outside world of the robot), it will be mandatory to solve this problem. That is because the robot will start with a specific configuration $\mathcal{S}_{q_0}$ and should move to another one $\mathcal{S}_{q_f}$. Nonetheless, the **trajectory generation problem** and the **control problem** should also be solved.

Finally, the problem of obtaining the position and orientation, in the Cartesian space, of the end effector of the hand (hand's end) when the generalized coordinates are known is called the **forward kinematics problem**. This problem will be studied in Section 3.

3 Forward kinematics of position and orientation

Every point P in the three-dimensional Cartesian space can be represented by coordinates related to any reference system. Moreover, two reference systems can relate to each other by means of a rotation of their coordinates axes and a translation of the coordinates origin. In Appendix A, those rotation matrices are defined and some properties are demonstrated. Those matrices will be used along the document.

Therefore, if the coordinates of a point P related to one reference system $\mathcal{S}_1$ are known, it is possible to obtain the coordinates related to another system, $\mathcal{S}_0$, if the rotation R_0^1 and translation t_0^1 matrices between both systems are known:

$$p_0 = R_0^1 p_1 + t_0^1 \tag{3.1}$$

where p_0 represents the coordinates of point P related to $\mathcal{S}_0$, p_1 represents the coordinates of the same point P related to $\mathcal{S}_1$ and t_0^1 represents the translation vector defined in $\mathcal{S}_0$. The transformation given by Equation 3.1, is called **rigid motion**.

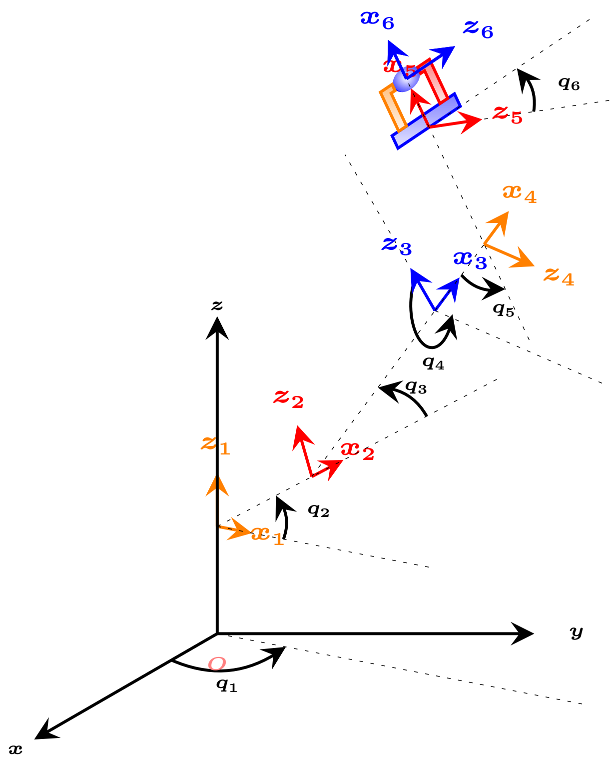


Figure 3.1: Local reference systems.

Figure 3.1 represents the local reference system $\mathcal{S}_i$ of every segment of the robot shown in Figure 2.1. Notice that only two axes have been represented. The third one is obtained by the cross product of the other two:

$$y_i = z_i \times x_i \tag{3.2}$$

As already mentioned, the election of the reference system on every segment is arbitrary. However, there exist systematic representations as the Denavit-Hartenberg representation, which will not be treated in this document. Nevertheless, because all joints of the robot are rotational, in this document the local reference systems will be defined as a rotation of the kinematic pair by keeping the name of the axis but modifying the subindex.

Let l_i be the distance between two joints of the kinematic pair $(i-1, i)$ and $(i, i+1)$. Distance l_0 is defined as the distance between the anchor to the ground (point O) and the joint of the kinematic pair $(0, 1)$. Therefore, the reference system $\mathcal{S}_0$ is a motionless reference system parallel to the inertial reference system (x, y, z). If there is not any confusion, $\mathcal{S}_0$ will be also used to represent the inertial reference system. On the other hand, because the kinematic pair $(n, n+1)$ does not exist, a significant point Q in the n-th segment will be defined. This point Q is called the "hand's end". This point is typically a tool when the robot is a manipulator, e.g. a gripper with two mobile fingers. In this example, the point Q could be the geometric center between the gripping point of the fingers. Therefore, distance l_n will be the distance between the kinematic pair $(n-1, n)$ and the point Q.

Moreover, let the set of rotation axes be defined as $\mathcal{U} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}, \vec{u_5}, \vec{u_6}\}$, where u_i represents the rotation of the segment related to its kinematic pair. Therefore, for the example of Figure 3.1 the set of rotation axes is:

$$\mathcal{U} = \{z_0, -y_1, -y_2, x_3, y_4, x_5\} \tag{3.3}$$

where $z_0 = z$.

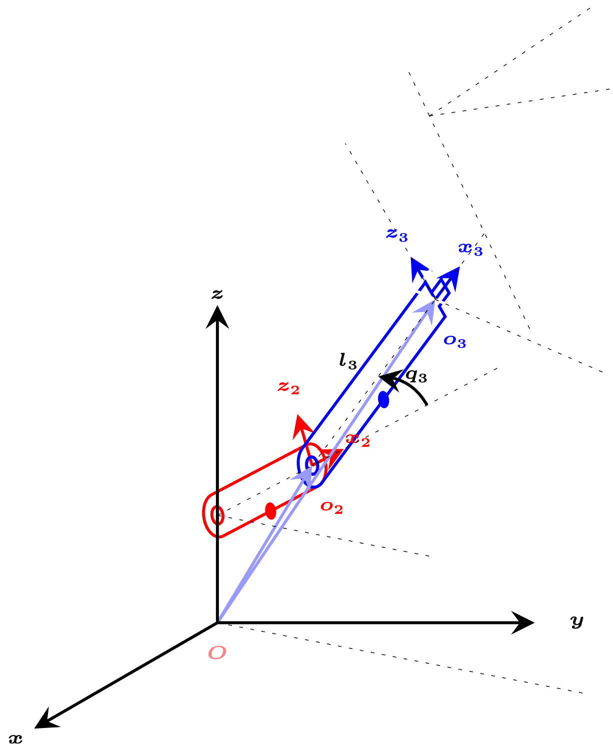


Figure 3.2: Local reference systems and kinematic pair $(2, 3)$.

The origin o_i of every local reference system $\mathcal{S}_i$ is located at the joint of the kinematic pair $(i, i+1)$. An exception exists with the origin o_n that is located at the already defined point Q. Figure 3.2 shows the kinematic pair $(2, 3)$ that allows to define the generalized coordinate q_3. As already defined, the rotation axis of segment 3 is $-y_2$. Moreover, according to the aforementioned notation, $l_i = |\overrightarrow{o_{i-1}o_i}|$.

Let the vector d_{i-1}^i, declared in the reference system $\mathcal{S}_{i-1}$, be defined as:

$$d_{i-1}^i = \overrightarrow{o_{i-1}o_i} \tag{3.4}$$

Moreover, let the set $\widetilde{\mathcal{O}}$ be defined as $\widetilde{\mathcal{O}} = \{\tilde{o}_0, \tilde{o}_1, \tilde{o}_2, \tilde{o}_3, \tilde{o}_4, \tilde{o}_5, \tilde{o}_6\}$, where $\tilde{o}_i$ refers to the point o_i declared in the reference system $\mathcal{S}_{i-1}$ prior to the rotation, that is for $q_i = 0$. Therefore, for the robot of Figure 3.1

$$\widetilde{\mathcal{O}} = \{l_0 z, l_1 z_0, l_2 x_1, l_3 x_2, l_4 x_3, l_5 x_4, l_6 x_5\} \tag{3.5}$$

According to this convention:

$$\boxed{d_{i-1}^i = R_{i-1}^i \tilde{o}_i} \tag{3.6}$$

For the example of the kinematic pair $(2, 3)$ of Figure 3.2

$$d_2^3 = R_2^3 \tilde{o}_3 \tag{3.7}$$

Segment 3 rotates around $\vec{u_3} = -y_2$ (see Equation 3.3), $\tilde{o}_3 = l_3 x_2$ (see Equation 3.5) and according to Appendix A:

$$R_2^3 = R_{-y,q_3} = \begin{bmatrix} \cos q_3 & 0 & -\sin q_3 \\ 0 & 1 & 0 \\ \sin q_3 & 0 & \cos q_3 \end{bmatrix} \tag{3.8}$$

Therefore, according to Equation 3.6:

$$d_2^3 = R_2^3 \tilde{o}_3 = \begin{bmatrix} \cos q_3 & 0 & -\sin q_3 \\ 0 & 1 & 0 \\ \sin q_3 & 0 & \cos q_3 \end{bmatrix} \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix} = l_3 \begin{bmatrix} \cos q_3 \\ 0 \\ \sin q_3 \end{bmatrix} \tag{3.9}$$

where $l_3 = |\overrightarrow{o_2 o_3}|$.

In what follows, some recursive relations for vectors d_i^j are derived. According to Equation 3.1, for any point P in the space:

$$p_0 = R_0^i p_i + d_0^i \tag{3.10a}$$
$$p_0 = R_0^{i-1} p_{i-1} + d_0^{i-1} \tag{3.10b}$$
$$p_{i-1} = R_{i-1}^i p_i + d_{i-1}^i \tag{3.10c}$$

where p_0 represents point P in coordinates of $\mathcal{S}_0$, p_i represents the same point P in coordinates of $\mathcal{S}_i$ and p_{i-1} represents the same point P in coordinates of $\mathcal{S}_{i-1}$.

Because the composition of rotations of local reference systems satisfies the chain rule, that is:

$$R_0^i = R_0^1 R_1^2 \cdots R_{i-1}^i \tag{3.11}$$

it will be satisfied, by substituting Equation 3.10c in 3.10b and subtracting Equation 3.10a, that

$$\boxed{d_0^i = d_0^{i-1} + R_0^{i-1} d_{i-1}^i} \tag{3.12}$$

Moreover, Equation 3.11 can be recursively expressed as:

$$\boxed{R_0^i = R_0^{i-1} R_{i-1}^i} \tag{3.13}$$

Equation 3.12 represents the **fundamental equation of the forward kinematics of position** and Equation 3.13 represents the **fundamental equation of the forward kinematics of orientation**.

Thanks to these two fundamental equations, the position of the final point of the hand $Q = o_n$ ($\overrightarrow{OQ} = l_0 \vec{k} + \overrightarrow{o_0 o_n} = l_0 \vec{k} + d_0^n$) and its orientation R_0^n can be calculated in the $\mathcal{S}_0$ reference system.

4 Hand orientation vector (a, n, s) from $\{\alpha_a, \alpha_s\}$ angles and a_z or s_z coordinates

It is common to define the orientation of the hand by three orthonormal vectors $\{a, n, s\}$, that will be called **approach**, **normal** and **sliding** respectively, where

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{4.1a}$$

$$n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{4.1b}$$

$$s = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \tag{4.1c}$$

Figure 4.1 shows the orientation of the hand in the space.

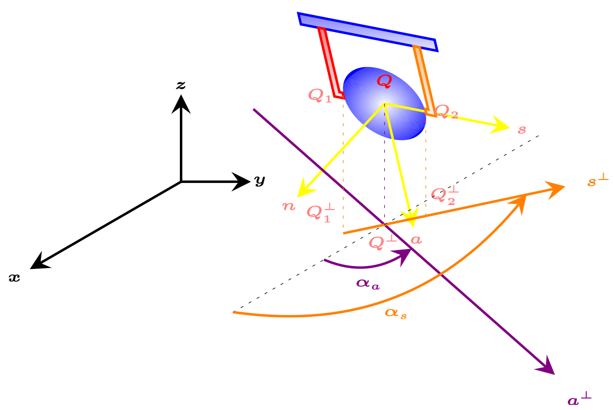


Figure 4.1: Hand orientation and projection over plane XY.

The orientation matrix $R_{\mathcal{S}_O}$ referred to the inertial system, can be represented in different forms, depending on the definition of the approach vector. For example, if the approach vector matches axis X, $R_{\mathcal{S}_O}$ will be represented by Equation 4.2a. On the other hand, if the approach vector matches axis Z, $R_{\mathcal{S}_O}$ will be represented by Equation 4.2b,

$$R_{\mathcal{S}_O} = \begin{bmatrix} a_x & -n_x & -s_x \\ a_y & -n_y & -s_y \\ a_z & -n_z & -s_z \end{bmatrix} \tag{4.2a}$$

$$R_{\mathcal{S}_O} = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} \tag{4.2b}$$

Nonetheless, both cases match the following relationships:

$$n = s \times a \tag{4.3a}$$

$$s = a \times n \tag{4.3b}$$

$$a = n \times s \tag{4.3c}$$

$$a \cdot n = a \cdot s \quad = n \cdot s = 0 \tag{4.3d}$$

where $\times$ represents the cross product and $\cdot$ the dot product.

The projections of the approach and sliding vectors over plane XY are not, in general, orthogonal, as shown in Figure 4.1.

Let the angles α_a and α_s be defined as:

$$a_x = |a^\perp| \cos\alpha_a \tag{4.4a}$$

$$a_y = |a^\perp| \sin\alpha_a \tag{4.4b}$$

$$s_x = |s^\perp| \cos\alpha_s \tag{4.4c}$$

$$s_y = |s^\perp| \sin\alpha_s \tag{4.4d}$$

where $|a^\perp|$ and $|s^\perp|$ represent the modules of the projections of approach and sliding vectors over the plane XY.

Given the approach vector a, it must be satisfied that $a \cdot s = 0$. Therefore, given angle α_s, the coordinate s_z can be obtained as follows:

$$s_z = -\frac{a_x}{a_z} s_x - \frac{a_y}{a_z} s_y \tag{4.5}$$

where $a_z \neq 0$.

If $a_z = 0$, then s_z can be any value, as for example $s_z = 0$.

Given the sliding vector s, a can be calculated similarly.

Because a and s are perpendicular vectors, it is easier to define for any value of $a'_z \neq 0$ or $s'_z \neq 0$, the direction vectors $a^\perp$ and $s^\perp$, (a'_x, a'_y) and (s'_x, s'_y) as:

$$a'_x = \cos\alpha_a \tag{4.6a}$$

$$a'_y = \sin\alpha_a \tag{4.6b}$$

$$s'_x = \cos\alpha_s \tag{4.6c}$$

$$s'_y = \sin\alpha_s \tag{4.6d}$$

$$a'_z = -\frac{s'_x}{s'_z} a'_x - \frac{s'_y}{s'_z} a'_y \tag{4.6e}$$

$$s'_z = -\frac{a'_x}{a'_z} s'_x - \frac{a'_y}{a'_z} s'_y \tag{4.6f}$$

Moreover, vectors $a' = (a'_x, a'_y, a'_z)$ and $s' = (s'_x, s'_y, s'_z)$ can be normalized to obtain a and s.

The modules of the projected vectors $a^\perp$ and $s^\perp$, are

$$|a^\perp| = \frac{1}{\sqrt{1 + (a'_z)^2}} \tag{4.7a}$$

$$|s^\perp| = \frac{1}{\sqrt{1 + (s'_z)^2}} \tag{4.7b}$$

The normal vector n will be obtained by the cross product of the sliding and approach vectors, $n = s \times a$.

A particular case, not contemplated in the previous equations, is when the hand is perpendicular to plane XY. In this case it is enough to calculate

$$a'_x = 0 \tag{4.8a}$$

$$a'_y = 0 \tag{4.8b}$$

and calculate s and n as previously mentioned.

5 Kinematic decoupling

Every 6 DOFs mechanism, in which its last three joints intersect in a point, allows the kinematic decoupling of position and orientation. The point of intersection of the last three joints is called **wrist center**. This means that it is possible to solve the inverse kinematics problems of position and orientation independently. More concretely, it is possible to obtain the generalized coordinates of the three first joints by solving an inverse kinematics problem of position and the last three ones by solving an inverse kinematics problem of orientation.

The mechanism studied in the previous sections satisfies this condition, because it has an spheric wrist. Therefore, the movement of the last three joints around their rotation axes does not change the wrist center.

Figure 5.1 shows this mechanism, where Q represents the position and $\{a, n, s\}$ the orientation of the hand's end.

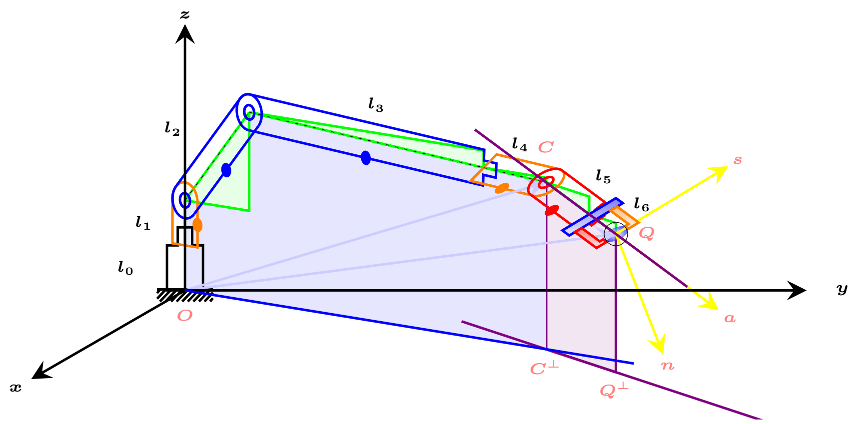


Figure 5.1: Kinematic decoupling.

The procedure must be implemented as follows:

1. Obtain the wrist center (C) when the approach vector ($\vec{a}$), the desired point of the hand's end (Q) and the hand dimension ($d_H = l_5 + l_6$) are known:

$$\overrightarrow{OC} = \overrightarrow{OQ} - d_M \ \vec{a} \tag{5.1}$$

2. Solve the inverse kinematics problem of position C to obtain $\{q_1, q_2, q_3\}$. This will be studied in Section 6.

3. Calculate the rotation matrix $R_0^3 = R_0^1 R_1^2 R_2^3$ that will be called $R_{\mathcal{S}_A}$.

4. Calculate $R_{\mathcal{S}_H}$ once the desired orientation of the mechanism $R_{\mathcal{S}_0}$ and $R_{\mathcal{S}_A}$ are known, as:

$$R_{\mathcal{S}_H} = R_{\mathcal{S}_A}^T R_{\mathcal{S}_0} \tag{5.2}$$

5. Solve the inverse kinematics problem of orientation R_3^6 to obtain $\{q_4, q_5, q_6\}$. This will be studied in Section 7.

6 Inverse kinematics of position

In this Section, a mechanism as the one shown in Figure 6.1, where C is the center of the wrist, is considered.

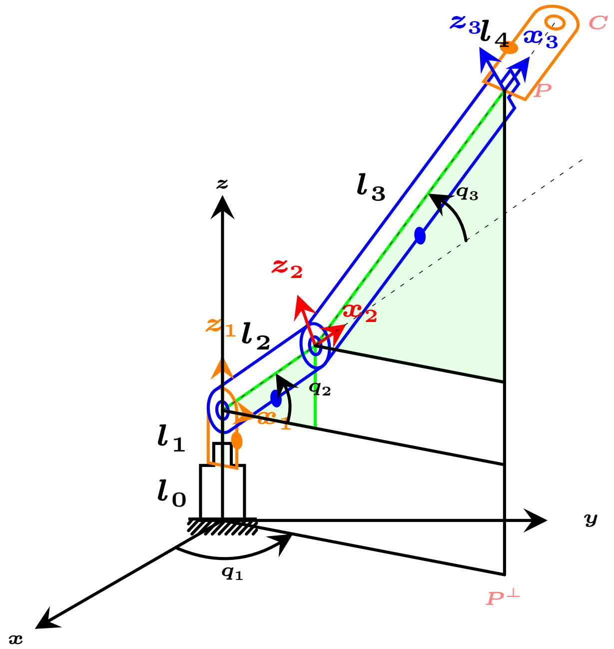


Figure 6.1: 4 DOFs mechanism.

The inverse kinematics problem consists of obtaining the generalized coordinates when the coordinates of a point Q, fixed in the hand's end, are known

$$Q = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix} \tag{6.1}$$

If the arm and hand can be decoupled, the approach axis can be located at a point C (wrist center) with coordinates

$$C = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \tag{6.2}$$

The wrist center C can be known if the dimensions of the segments of the hand (l_5 and l_6) and the approach vector a are known,

$$\boxed{\overrightarrow{OC} = \overrightarrow{OQ} - (l_5 + l_6)\ \vec{a}} \tag{6.3}$$

where O is the coordinates origin of the reference system $\mathcal{S}_O$.

The approach vector can be obtained from the rotation matrix of the fixed system at the hand's end ($R_{\mathcal{S}_O}$). For example, if this matrix has been defined as in Equation 4.2a, then

$$a = \begin{bmatrix} a_x & -n_x & -s_x \\ a_y & -n_y & -s_y \\ a_z & -n_z & -s_z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6.4}$$

A compact form of writing coordinates of point C is

$$\overrightarrow{OC} = \overrightarrow{OQ} - (l_5 + l_6) R_{\mathcal{S}_O} \vec{i} \tag{6.5}$$

where

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6.6}$$

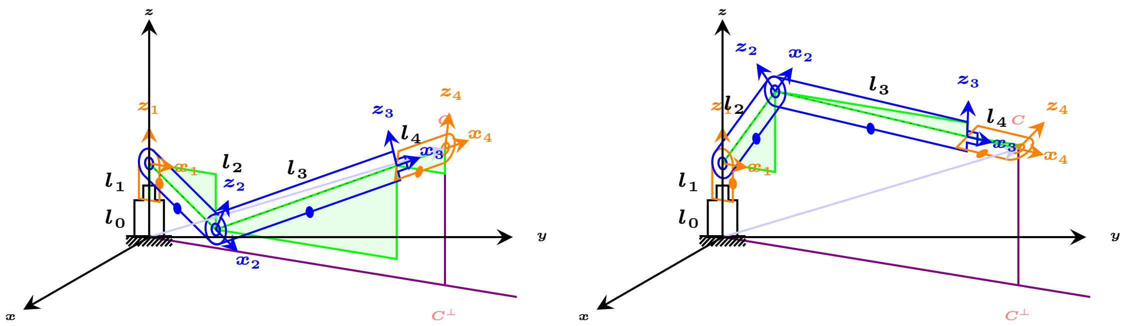
If this matrix was defined as in Equation 4.2b, that is, the approach axis was axis Z,

$$\overrightarrow{OC} = \overrightarrow{OQ} - (l_5 + l_6) R_{\mathcal{S}_O} \vec{k} \tag{6.7}$$

where

$$\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.8}$$

The inverse kinematics problem of position consist of obtaining the generalized coordinates $\{q_1, q_2, q_3\}$ when the coordinates of point C are known. Figure 6.2 shows the two possible solutions for a similar example.



(a) $R^{I}_{\mathcal{S}_A}$ configuration. (b) $R^{II}_{\mathcal{S}_A}$ configuration.

Figure 6.2: Two different solutions to the same inverse kinematics problem of position.

Appendix B shows how to obtain the different solutions $\{q_1, q_2, q_3\}$ when the arm rotation axes are $\{z_0, -y_1, -y_2\}$,

$$q_1 = \begin{cases} atan2\left(C_y, C_x\right) & C_x \neq 0 \vee C_y \neq 0 \\ \mathbb{R} & C_x = C_y = 0 \end{cases} \tag{6.9a}$$

$$q_2 = -\beta + atan2\left(L, \pm\sqrt{(1 - L^2)}\right) \tag{6.9b}$$

$$q_3 = -q_2 + atan2\left(C_{z'} - l_2 \sin q_2, C_{x'} - l_2 \cos q_2\right) \tag{6.9c}$$

where

$$\beta = atan2\left(C_{z'}, C_{x'}\right) \tag{6.10a}$$

$$L = \frac{l_2^2 + C_{x'}^2 + C_{z'}^2 - (l_3 + l_4)^2}{2 l_2 \sqrt{\left(C_{x'}^2 + C_{z'}^2\right)}} \tag{6.10b}$$

$$C_{x'} = C_x \cos q_1 + C_y \sin q_1 \tag{6.10c}$$

$$C_{z'} = C_z - (l_0 + l_1) \tag{6.10d}$$

7 Inverse kinematics of orientation

The inverse kinematics problem of orientation consists of obtaining the generalized coordinates when the rotation matrix $\mathcal{S}_O$ of a fix reference system at the hand's end is known.

$$R_{\mathcal{S}_O} = R_0^3 R_3^6 \tag{7.1}$$

If the arm and hand can be decoupled, $R_{\mathcal{S}_A} = R_0^3$ is known if the inverse kinematics problem of position is already solved. As a consequence, $R_{\mathcal{S}_H} = R_3^6$ can be obtained by solving:

$$R_{\mathcal{S}_H} = R_{\mathcal{S}_A}^T R_{\mathcal{S}_O} \tag{7.2}$$

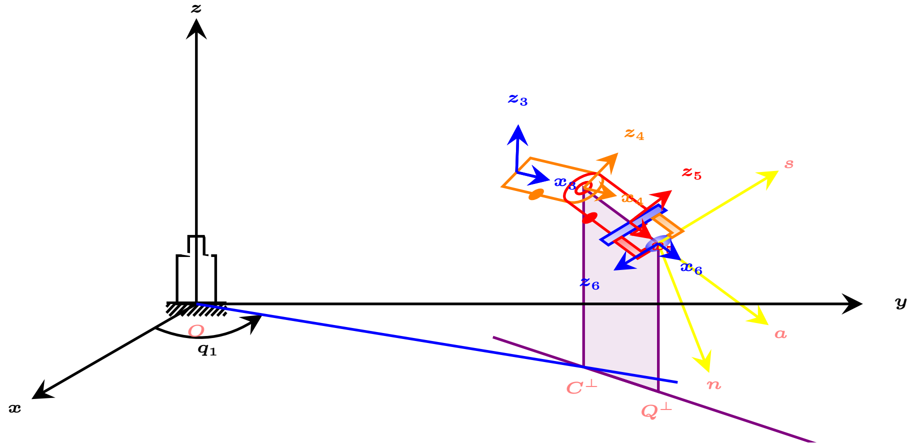
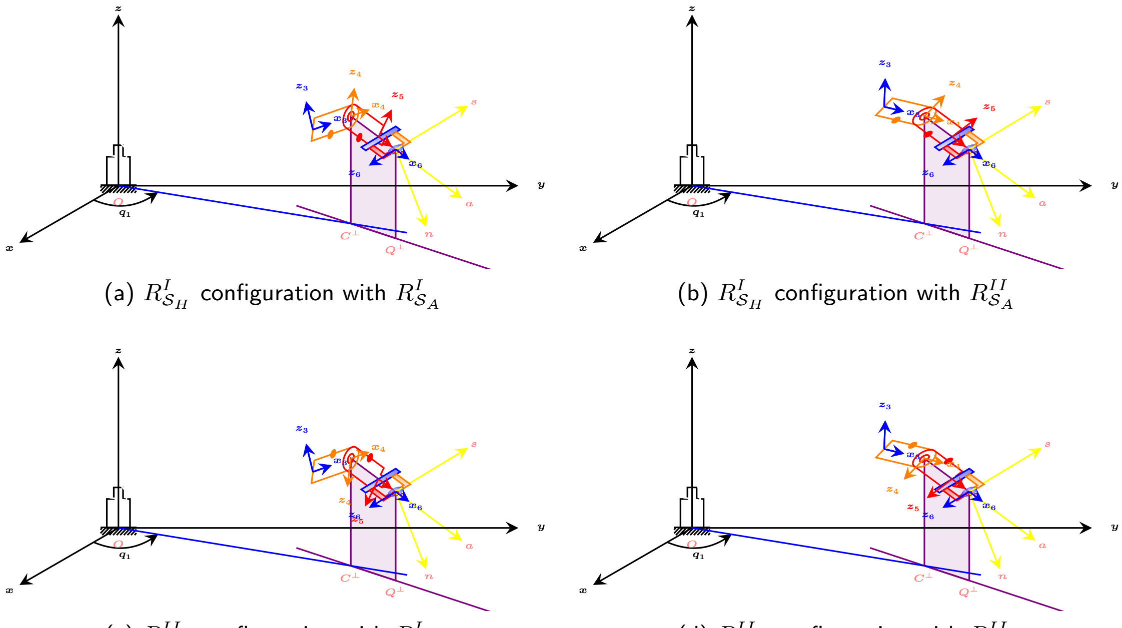


Figure 7.1: $R^{I}_{\mathcal{S}_H}$ configuration with $R^{II}_{\mathcal{S}_A}$.

Figure 7.1 shows one of the possible solutions to the inverse kinematics problem of orientation for the solution $R^{II}_{\mathcal{S}_A}$ of the inverse kinematics problem of position obtained in Section 6.



(a) $R^{I}_{\mathcal{S}_H}$ configuration with $R^{I}_{\mathcal{S}_A}$

(b) $R^{I}_{\mathcal{S}_H}$ configuration with $R^{II}_{\mathcal{S}_A}$

(c) $R^{II}_{\mathcal{S}_H}$ configuration with $R^{I}_{\mathcal{S}_A}$

(d) $R^{II}_{\mathcal{S}_H}$ configuration with $R^{II}_{\mathcal{S}_A}$

Figure 7.2: Solutions to the inverse kinematics problem of orientation.

Figure 7.2 shows the two possible solutions to the inverse kinematics problem of orientation for every solution to the inverse kinematics problem of position obtained in Section 6.

The rotation matrix R^6_3 is obtained as a composition of three successive rotations. Therefore, it has three unknowns variables which represent the generalized coordinates $\{q_4, q_5, q_6\}$ of the hand. That is:

$$R^6_3 = R^4_3 R^5_4 R^6_5 \tag{7.3}$$

The generalized coordinates $\{q_4, q_5, q_6\}$ chosen represent the rotation angles relatives to the prior segment. Let the rotation axes be $\{x_3, y_4, x_5\}$. As a consequence:

$$R^4_3 = R_{X,q_4} \tag{7.4a}$$
$$R^5_4 = R_{Y,q_5} \tag{7.4b}$$
$$R^6_5 = R_{X,q_6} \tag{7.4c}$$

where R_{X,q_4}, R_{Y,q_5} and R_{X,q_6} represent the elemental rotation matrix around axis X, Y and X respectively.

$$R_3^6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_4 & -\sin q_4 \\ 0 & \sin q_4 & \cos q_4 \end{bmatrix} \begin{bmatrix} \cos q_5 & 0 & \sin q_5 \\ 0 & 1 & 0 \\ -\sin q_5 & 0 & \cos q_5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_6 & -\sin q_6 \\ 0 & \sin q_6 & \cos q_6 \end{bmatrix} \tag{7.5}$$

By developing Equation 7.5,

$$R_3^6 = \begin{bmatrix} c_5 & s_5 s_6 & s_5 c_6 \\ s_4 s_5 & c_4 c_6 - s_4 c_5 s_6 & -c_4 s_6 - s_4 c_5 c_6 \\ -c_4 s_5 & s_4 c_6 + c_4 c_5 s_6 & -s_4 s_6 + c_4 c_5 c_6 \end{bmatrix} = U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \tag{7.6}$$

A specific case occurs when $s_5 = 0$. In this case, matrix R_3^6 remains as one of the following equations:

$$R_3^6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_4 c_6 - s_4 s_6 & -c_4 s_6 - s_4 c_6 \\ 0 & s_4 c_6 + c_4 s_6 & -s_4 s_6 + c_4 c_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(q_4 + q_6) & -\sin(q_4 + q_6) \\ 0 & \sin(q_4 + q_6) & \cos(q_4 + q_6) \end{bmatrix} \tag{7.7a}$$

$$R_3^6 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & c_4 c_6 + s_4 s_6 & -c_4 s_6 + s_4 c_6 \\ 0 & s_4 c_6 - c_4 s_6 & -s_4 s_6 - c_4 c_6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(q_4 - q_6) & \sin(q_4 - q_6) \\ 0 & \sin(q_4 - q_6) & -\cos(q_4 - q_6) \end{bmatrix} \tag{7.7b}$$

Therefore, the inverse kinematics problem of orientation consists of solving the nine non-linear equations that can be obtained from identifying the following matrices:

$$R_3^6 = R_{\mathcal{S}_H} \tag{7.8}$$

However, because they are rotation matrices, it will only be necessary to solve three independent equations of the identification. Therefore, solutions can be written as follows:

$$q_5 = atan2\left(\pm\sqrt{1 - u_{11}^2}, u_{11}\right) \tag{7.9a}$$

$$q_4 = \begin{cases} atan2\,(u_{21}, -u_{31}) & , \sin q_5 > 0 \\ atan2\,(-u_{21}, u_{31}) & , \sin q_5 < 0 \\ -u_{11} q_6 + atan2\,(u_{32}, u_{22}) & , |u_{11}| = 1 \end{cases} \tag{7.9b}$$

$$q_6 = \begin{cases} atan2\,(u_{12}, u_{13}) & , \sin q_5 > 0 \\ atan2\,(-u_{12}, -u_{13}) & , \sin q_5 < 0 \\ u_{11}\,(-q_4 + atan2\,(u_{32}, u_{22})) & , |u_{11}| = 1 \end{cases} \tag{7.9c}$$

It can be observed that two possible solutions are obtained when $|u_{11}| \neq 1$. Moreover, infinite solutions can be found when $|u_{11}| = 1$, because in this case there is a dependence between q_4 and q_6.

8 Kinematic conditions for $q_4 = 0$: $\{\alpha_Q, \Delta_4\}$ angles

This Section shows the conditions to be met when $q_4 = 0$, assuming that the arm and hand can be decoupled. To impose $q_4 = 0$ is equivalent to impose that the approach vector a belongs to the movement plane of the arm. Therefore, $Q^\perp$ belongs to the straight line that cross the origin and $C^\perp$. Moreover, this straight line will have as direction vector the projection $a^\perp$ of the approach vector a over the XY plane.

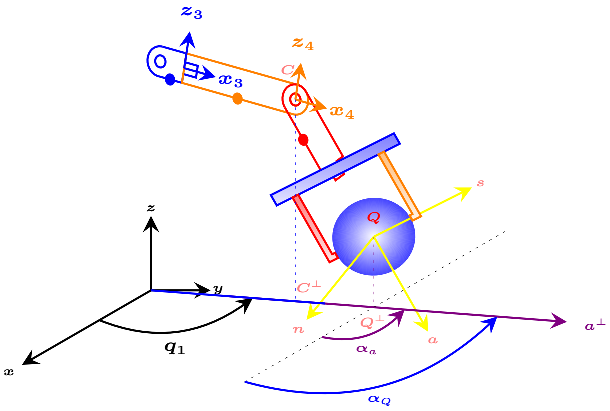


Figure 8.1: Hand orientation with $q_4 = 0$.

Figure 8.2 shows a situation in which $q_4 = 0$.
Let Δ_4 be a phase lag defined when $q_4 \neq 0$ in such a way that $\Delta_4 = 0$ if and only if $q_4 = 0$.

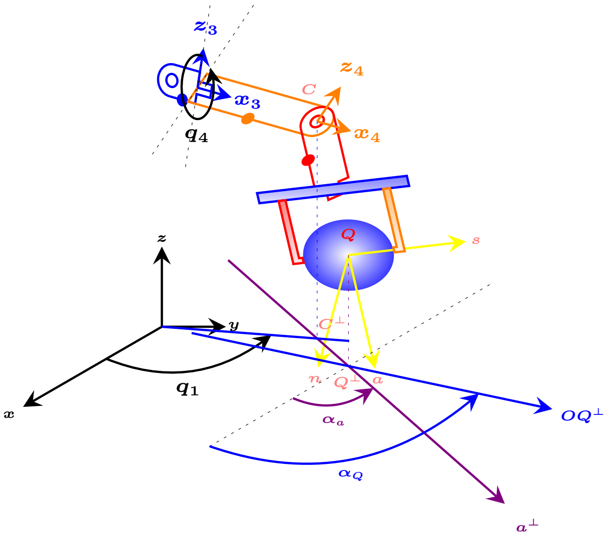


Figure 8.2: Hand orientation when $q_4 = -\pi/4$.

It is always true that the straight line passing through $C^\perp = (C_x, C_y, 0)$ and $Q^\perp = (Q_x, Q_y, 0)$ has a director vector $a^\perp = (a_x, a_y, 0)$.

Let α_Q be the angle between the straight line that cross the origin and $Q^\perp$, and α_a the angle between $a^\perp$ and the X axis. Let Δ_4 be defined as

$$\Delta_4 = \alpha_Q - \alpha_a \tag{8.1}$$

Equation 8.1 satisfies the condition that $\Delta_4 = 0$ if and only if $q_4 = 0$. Moreover, $\Delta_4 = 0$ if and only if $q_1 = \alpha_Q$.

According to these definitions, it should be fulfilled that

$$a_x = |a^\perp| \cos \alpha_a = |a^\perp| \cos(\alpha_Q - \Delta_4) \tag{8.2a}$$
$$a_y = |a^\perp| \sin \alpha_a = |a^\perp| \sin(\alpha_Q - \Delta_4) \tag{8.2b}$$

where

$$Q_x = |Q^\perp| \cos \alpha_Q \tag{8.3a}$$
$$Q_y = |Q^\perp| \sin \alpha_Q \tag{8.3b}$$

In consequence, given any point Q, any a_z value and obtained a_x and a_y through Equation 8.2 with $\Delta_4 = 0$, it warranties that $q_4 = 0$. Any value $\Delta_4 \neq 0$ will make $q_4 \neq 0$.

R_3^6 given by Equation 7.6 when $q_4 = 0$ is converted to:

$$R_3^6 = \begin{bmatrix} c_5 & s_5 s_6 & s_5 c_6 \\ 0 & c_6 & -s_6 \\ -s_5 & c_5 s_6 & c_5 c_6 \end{bmatrix} = U \tag{8.4}$$

It can be observed that it is immediate to obtain the generalized coordinates $\{q_5, q_6\}$,

$$q_5 = atan2\left(-u_{31}, u_{11}\right) \tag{8.5a}$$
$$q_6 = atan2\left(-u_{23}, u_{22}\right) \tag{8.5b}$$

Different from the general case where $q_4 \neq 0$, now there is a unique solution to the inverse kinematics problem of orientation. Moreover, the approach vector a will be limited to the plane defined by q_1.

A Vector operations and rotation matrices

Definition A.1 (Orthogonal matrix R). *An squared orthogonal matrix R is a matrix which rows and columns form an orthonormal base of an Euclidean space, that is, their module is one and are orthogonal between them.*

The inverse of an orthogonal matrix is its transpose matrix,

$$R^T R = I \tag{A.1}$$

where I is the identity matrix and R^T is the transpose matrix of R.

The determinant of an orthogonal matrix is $det(R) = 1$ or $det(R) = -1$, but not every matrix with a determinant 1 or -1 is orthogonal.

Definition A.2 (Elemental rotation matrices R_x, R_y, R_z). *Rotation matrices $R \in SO(n)$, where the set $SO(n)$ is a special orthogonal group of dimension n. In general, the dimension of $SO(n)$ is $\dfrac{n(n-1)}{2}$.*

The set $SO(3)$ is of dimension 3, that means that it is always possible to transform a reference system into another one by a consecutive sequence of three rotations around a single axis (elementary rotations).

The set $SO(2)$ is of dimension 1, that means that it is always possible to transform a reference system into another one by a single rotation around a single axis.

Let the elementary rotation matrices $R \in SO(3)$ around axis x, y and z be defined as:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{A.2a}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{A.2b}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.2c}$$

The elementary rotation matrix $R \in SO(2)$ is defined as:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{A.3}$$

Proposition A.3 (Rotation composition). *Let the matrix R_i^j be defined as the rotation matrix of the reference system $\mathcal{S}_j$ with respect to $\mathcal{S}_i$.*

Given the rotation matrices R_i^k and R_k^j and the reference systems $\mathcal{S}_i$, $\mathcal{S}_j$ y $\mathcal{S}_k$, it is true that

$$R_i^j = R_i^k R_k^j \tag{A.4}$$

For example, R_0^3 can be rewritten by following the chain rule as:

$$R_0^3 = R_0^1 R_1^2 R_2^3 \tag{A.5}$$

Proposition A.4 (Coordinates of a point in two reference systems). *Let point P be defined as a fixed point in the space which coordinates in the reference system $\mathcal{S}_i$ are given by vector p_i and let matrix R_i^j be defined as the rotation matrix of the reference system $\mathcal{S}_j$ with respect to $\mathcal{S}_i$. Then it is fulfilled that*

$$p_i = R_i^j p_j \tag{A.6}$$

For example, if R_i^j is the elemental matrix $R_\theta \in SO(2)$, then if $p_j = (\cos\theta, -\sin\theta)^T$, it is true that $p_i = (1, 0)^T$. It can be observed that $p_j = (R_i^j)^T p_i = R_j^i p_i$.

Proposition A.5 (Antisymmetric matrix S). *Given two vectors $a, b \in \mathbb{R}^3$, there is a single antisymmetric matrix $S(a)$ such that*

$$a \times b = S(a)b \tag{A.7}$$

where $\times$ represents the cross product and

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{A.8}$$

Antisymmetric matrices S satisfy the condition that $S + S^T = O$.

Proposition A.6. *Because $a \times b = -b \times a$ it is fulfilled that*

$$a \times b = -S(b)a \tag{A.9}$$

Proposition A.7. *For every rotation matrix $R \in SO(3)$*

$$R(a \times b) = (Ra) \times (Rb) = S(Ra)Rb = -S(Rb)Ra \tag{A.10}$$

Proposition A.8. *For every rotation matrix $R \in SO(3)$*

$$S(Ra)Rb = RS(a)b \tag{A.11}$$

Proposition A.9. *By substituting $c = Rb$ in A.11, for every rotation matrix $R \in SO(3)$, it is true that*

$$RS(a)R^T c = S(Ra)c \tag{A.12}$$

Proposition A.10. *Because $R^T R = I$, the square of the Euclidean norm of vector a can be rewritten as:*

$$a \cdot a = a^T a = a^T R^T Ra = (Ra)^T (Ra) = (Ra) \cdot (Ra) \tag{A.13}$$

where $\cdot$ is the dot product.

This means that the norm is invariant to rotations. In general, it can be assured that

$$a \cdot b = (Ra) \cdot (Rb) \tag{A.14}$$

where R is an orthogonal matrix.

Proposition A.11.

$$a \cdot b = tr(ab^T) = \frac{1}{2} tr[S(a)^T S(b)] \tag{A.15}$$

where $tr[A]$ represents the trace of matrix A.

The norm of Frobenius of matrix A is defined as $||A||_F = \sqrt{tr[AA^T]}$.

Proposition A.12. *Given a rotation matrix $R(t)$ it is true that:*

$$\dot{R}(t) = S(w(t))R(t) \tag{A.16}$$

where $w(t)$ represents the angular speed vector and the antisymmetric matrix $S(w(t))$, defined as Equation A.8, is called **angular speed tensor***.*

Proposition A.12 demonstration. It is known that $R^T R = RR^T = I$.

It can be said that

$$\dot{R}(t) = \dot{R}(t)R^T R \tag{A.17}$$

It can be demonstrated that $S = \dot{R}(t)R^T$ is an antisymmetric matrix.

By deriving expression $RR^T = I$,

$$\dot{R}R^T + R\dot{R}^T = O \tag{A.18}$$

that is $S + S^T = O$, hence S is an antisymmetric matrix. □

B Solution to the inverse kinematics of position

In this Section the equations of the inverse kinematics problem of position are solved by knowing the wrist's center C of the mechanism of Figure B.1.

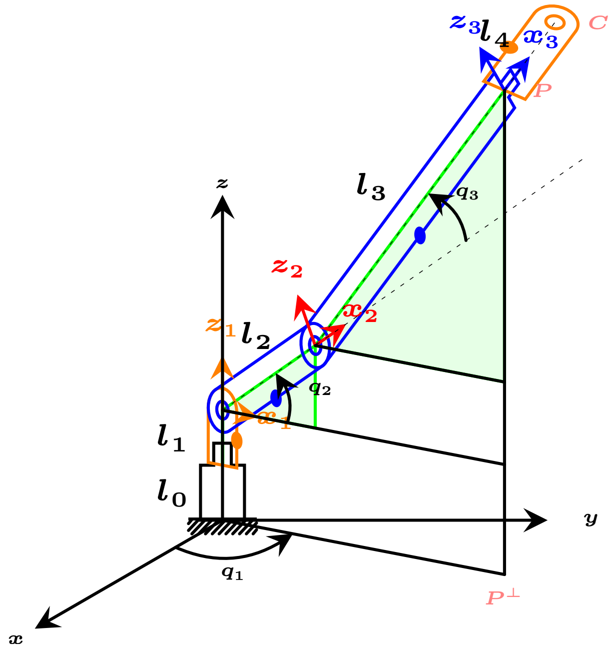


Figure B.1: 4 DOFs robot.

Let the rotation axes of the robot be $\{z_0, -y_1, -y_2, x_3\}$. Therefore, because x_3 is the rotation axis of the fourth segment, the mechanism is equivalent, from the solution of the inverse kinematics problem of position perspective, to other mechanism of 3 DOFs in which the length of the last segment is $l_3 + l_4$. That means that the rotation around x_3 does not affect to the solution of the inverse kinematics problem of position.

Because the rotation axis of the first segment is z_0, q_1 can be immediately obtained as:

$$q_1 = atan2\left(C_y, C_x\right) \tag{B.1}$$

To obtain q_2 and q_3 it is mandatory to solve the following trigonometric equations' system:

$$C_{x'} = l_2 \cos q_2 + (l_3 + l_4)\cos\left(q_2 + q_3\right) \tag{B.2a}$$
$$C_{z'} = l_2 \sin q_2 + (l_3 + l_4)\sin\left(q_2 + q_3\right) \tag{B.2b}$$

where

$$C_x = C_{x'} \cos q_1 \tag{B.3a}$$
$$C_y = C_{x'} \sin q_1 \tag{B.3b}$$
$$C_z = C_{z'} + l_0 + l_1 \tag{B.3c}$$
$$\tag{B.3d}$$

The solution of $C_{x'}$ can be obtained from Equation B.3a and Equation B.3b:

$$C_{x'} = C_x \cos q_1 + C_y \sin q_1 \tag{B.4}$$

The equations' system of Equation B.2 has two solutions that can be obtained by using:

$$\left(C_{x'} - l_2 \cos q_2\right)^2 + \left(C_{z'} - l_2 \sin q_2\right)^2 = \left(l_3 + l_4\right)^2 \tag{B.5}$$

By simplifying Equation B.5:

$$C_{x'} \cos q_2 + C_{z'} \sin q_2 = \frac{l_2^2 + C_{x'}^2 + C_{z'}^2 - (l_3 + l_4)^2}{2 l_2} \tag{B.6}$$

To solve Equation B.6, the expression of the sin of the angles sum can be used; therefore:

$$\sin(q_2+\beta)=\frac{l_2^2+C_{x'}^2+C_{z'}^2-(l_3+l_4)^2}{2l_2\sqrt{(C_{x'}^2+C_{z'}^2)}} \tag{B.7}$$

where

$$\sin\beta=\frac{C_{x'}}{\sqrt{(C_{x'}^2+C_{z'}^2)}} \tag{B.8a}$$

$$\cos\beta=\frac{C_{z'}}{\sqrt{(C_{x'}^2+C_{z'}^2)}} \tag{B.8b}$$

and in computational form:

$$q_2=-\beta+atan2\left(L,\pm\sqrt{(1-L^2)}\right) \tag{B.9}$$

where

$$\beta=atan2\left(C_{x'},C_{z'}\right) \tag{B.10a}$$

$$L=\frac{l_2^2+C_{x'}^2+C_{z'}^2-(l_3+l_4)^2}{2l_2\sqrt{(C_{x'}^2+C_{z'}^2)}} \tag{B.10b}$$

To obtain q_3 it is enough to solve the equations' system given by Equation B.2, when q_2 is known; that is:

$$q_3=-q_2+atan2\left(C_{z'}-l_2\sin q_2,C_{x'}-l_2\cos q_2\right) \tag{B.11}$$

As shown in Equation B.9, the inverse kinematics problem of position has two possible solutions, although there are singular conditions when $C_x=C_y=0$, because Equation B.1 can not be applied. In this situation, q_1 could have any value. From Equation B.4 it can be obtained that if $C_x=C_y=0$ then $C_{x'}=0$ and vice-versa. Moreover, if $C_{x'}=0$, then $\sin\beta=0$ so $\beta=0$ or $\beta=\pi$ obtaining two possible solutions for q_2 from Equation B.1, that is , $\sin q_2=\pm L$. Nonetheless $\cos\beta=\pm 1$, what implies that $C_{z'}=\pm 1$.

It can also be observed that if $C_{x'}=C_{z'}=0$, β cannot be obtained from Equation B.10a. In this case, it is mandatory that $l_2=l_3+l_4$, as can be checked from Equation B.5. This singularity can be easily solved from Equation B.2, as:

$$0=l_2\cos q_2+(l_3+l_4)\cos(q_2+q_3) \tag{B.12a}$$

$$0=l_2\sin q_2+(l_3+l_4)\sin(q_2+q_3) \tag{B.12b}$$

(B.12c)

By multiplying Equation B.12a by $\sin q_2$ and Equation B.12b by $\cos q_2$ and subtracting one from the other, it can be obtained that $\sin q_3=0$, so $q_3=0$ or $q_3=\pi$. However, $q_3=0$ does not satisfies Equations B.12, but $q_3=\pi$ does it for any value of q_2.

However, when $l_2=l_3+l_4$ it is not necessary that $C_{x'}=C_{z'}=0$. If $C_{x'}\neq 0$ or $C_{z'}\neq 0$ there is not this singularity and β could be obtained uniquely. Therefore, if the first singularity $C_x=C_y=0$ is not meet, neither is the second one. But if the first one is meet and $C_z=l_0+l_1$, the second singularity will be meet. From the robot design point of view, it will be interesting that $l_2\neq l_3+l_4$, and the singularity will be avoided.

Finally, it can also occur that $C_{x'}=l_2\cos q_2$ and $C_{z'}=l_2\sin q_2$, so q_2 cannot be obtained form Equation B.9, because $l_2=\sqrt{(C_{x'}^2+C_{z'}^2)}$. Moreover, because Equation B.5, $l_3+l_4=0$ what is impossible except for $l_3=l_4=0$. Therefore, this singularity cannot be produced because $l_3>0$, although it could be considered $l_4=0$ without loss of generalization. This implies that it will always be possible to obtain q_3 given q_2 and using Equation B.12.

In summary, if $l_2\neq l_3+l_4$, there is only the possibility of the first singularity.

To conclude, and assuming that $l_2 \neq l_3 + l_4$, the inverse kinematics problem of position has the following solutions:

$$\boxed{\begin{aligned} q_1 &= \begin{cases} atan2\left(C_y, C_x\right) & C_x \neq 0 \vee C_y \neq 0 \\ \mathbb{R} & C_x = C_y = 0 \end{cases} && \text{(B.13a)} \\ q_2 &= -\beta + atan2\left(L, \pm\sqrt{(1-L^2)}\right) && \text{(B.13b)} \\ q_3 &= -q_2 + atan2\left(C_{z'} - l_2 \sin q_2, C_{x'} - l_2 \cos q_2\right) && \text{(B.13c)} \end{aligned}}$$

where

$$\boxed{\begin{aligned} \beta &= atan2\left(C_{z'}, C_{x'}\right) && \text{(B.14a)} \\ L &= \frac{l_2^2 + C_{x'}^2 + C_{z'}^2 - (l_3 + l_4)^2}{2 l_2 \sqrt{\left(C_{x'}^2 + C_{z'}^2\right)}} && \text{(B.14b)} \\ C_{x'} &= C_x \cos q_1 + C_y \sin q_1 && \text{(B.14c)} \\ C_{z'} &= C_z - (l_0 + l_1) && \text{(B.14d)} \end{aligned}}$$