Introduction to Haptic Systems

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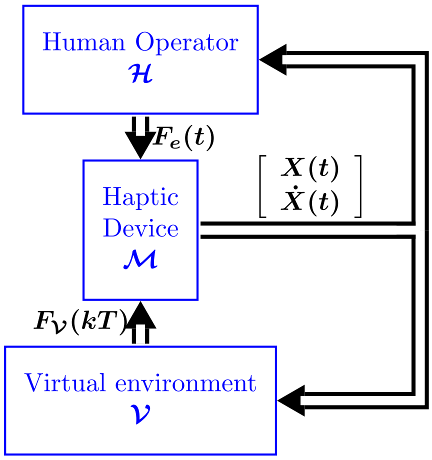

Figure 1.1: Block diagram of a haptic system.

The objective of this document is to introduce haptic systems. The dynamic equations of their more important components will be developed and some characteristics of their operation will be demonstrated. Figure 1.1 shows the different components of a haptic system and their relationship, where $F_e(t)$ and $F_{\mathcal{V}}(kT)$ represent variable forces with respect to the continuous and discrete time, respectively and $X(t), \dot{X}(t)$ represent the position and velocity of the end of the haptic system or robot.

The study is limited to robots of one DOF, made up of a DC motor and a rigid load attached by means of a reduction gearbox. Section 2 presents the dynamic equations of a DC motor and Section 3 the dynamic equation of the load by means of the Euler-Lagrange formulation and its integration in the simplified equation of the DC motor.

In Section 4 the equations of the effect of applying an external force to the end of the robot, in the case that there is not haptic feedback, are obtained. If there is not haptic feedback, there is not a closed-loop between the haptic device and the human operator, nor a virtual environment, as shown in Figure 1.2. Under these circumstances, a position control system is developed in Section 5 and a gravity torque compensator control system in Section 6. The main idea is to show how to design a control system that allows i) that the human operator can move the end of the haptic system by applying a force and ii) that the haptic system stops moving when no force is applied. The control technique used is shown in Section 5. The main reason to develop this technique is that it allows to conduct some studies as if the haptic system were lineal. This eases the analysis and allows to visualize, in a simple way, some characteristics of haptic systems of more than 1 DOF.

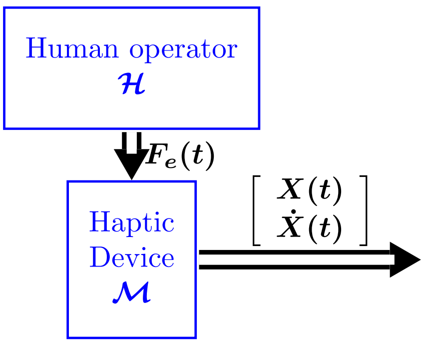

Figure 1.2: Force applied to the end of the haptic system.

Up to that point, Figure 1.2 is completely explained, where the force applied by the human operator does not depend on the haptic device movement, that is, there is not haptic feedback. In [1] this study is completely developed, including hands with robotic fingers and focusing on the real objects grasping problem.

Later on, Section 7 introduces the concept of haptic feedback and a virtual environment with virtual objects exerting forces over the DC motor of the haptic device. In Section 8 a mathematical model in discrete time of a simple virtual object, a torsion spring with viscous friction, is developed.

Finally, Section 9 incorporates all the control elements previously studied, emphasizing the haptic stability and demonstrating why oscillations can appear at the end of the haptic robot.

Therefore, the main idea of the document is to introduce step by step the design of a haptic system, although limited to a 1 DOF robot. The extension to haptic devices of more DOFs does not substantially change what is developed here, except for the fact that the dynamic equations are more complex, and therefore the analysis requires more sophisticated techniques.

2 DC Motor

Figure 2.1 represents a DC motor.

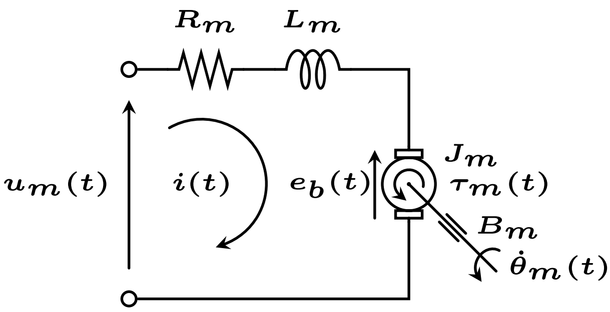

Figure 2.1: DC motor.

The electric equation of the motor is

$$u_m(t) = R_m i(t) + L_m \frac{di(t)}{dt} + e_b(t) \tag{2.1}$$

where $u_m(t)$ represents the input voltage, $i(t)$ the electric current, $e_b(t)$ the counter-electromotive force (back-EMF), R_m the terminal resistance and L_m the robot inductance.

The mechanical equation of the motor is

$$\tau_m(t) = J_m \ddot{\theta}_m(t) + \tau_l(t) + \tau_f(t) \tag{2.2}$$

where $\tau_m(t)$ represents the motor torque, $\ddot{\theta}_m(t)$ the angular acceleration of the motor, J_m the rotor inertia, $\tau_l(t)$ the load torque from the motor axis and $\tau_f(t)$ the friction torque which considers different friction components such as the damping one amongst others.

In what follows, only the damping viscous torque will be taken into account, therefore

$$\tau_m(t) = J_m \ddot{\theta}_m(t) + B_m \dot{\theta}_m(t) + \tau_l(t) \tag{2.3}$$

where $\dot{\theta}_m(t)$ is the angular motor velocity, $B_m \dot{\theta}_m(t)$ the damping viscous torque with B_m the damping viscous constant.

Typically, a DC motor satisfies the following electromechanical coupling equations:

$$\boxed{\begin{aligned} e_b(t) &= k_b \dot{\theta}_m(t) && \text{(2.4a)} \\ \tau_m(t) &= k_m i(t) && \text{(2.4b)} \end{aligned}}$$

where k_b and k_m are motor constants, back-EMF constant and torque constant, respectively. When they are expressed in the same unit system, $k_b = k_m$.

Let the **electric time constant** t_e and the **mechanical time constant** t_m (whose units in the international system are in seconds), be defined as:

$$t_e = \frac{L_m}{R_m} \tag{2.5a}$$

$$t_m = \frac{R_m J_m}{R_m B_m + k_b k_m} \tag{2.5b}$$

The DC motor equations can be simplified taking into account that the electric constant of a DC motor is usually much smaller than its mechanical constant, that is, $t_e \ll t_m$. This can be interpreted as neglecting the inductance L_m contribution in the electric equation. Therefore, the electric equation given by Equation 2.1 can be rewritten as:

$$u_m(t) = R_m i(t) + k_b \dot{\theta}_m(t) \tag{2.6}$$

By solving the mechanical equation (Equation 2.3) for $i(t)$, taking into account Equation 2.4b and substituting in Equation 2.6, the **simplified equation of a DC motor** is obtained

$$u_m(t) = \frac{R_m J_m}{k_m}\ddot{\theta}_m(t) + \left(\frac{R_m B_m}{k_m} + k_b\right)\dot{\theta}_m(t) + \frac{R_m}{k_m}\tau_l(t) \tag{2.7}$$

Equation 2.7 represents, in the absence of load, a first order linear differential equation for the variable of angular velocity and a second order linear differential equation for the variable of angular position.

It will be relevant to write the simplified equation in terms of the torque in the motor axis, instead of electrical voltages

$$\boxed{\frac{k_m}{R_m}u_m(t) = J_m\ddot{\theta}_m(t) + \left(B_m + \frac{k_b k_m}{R_m}\right)\dot{\theta}_m(t) + \tau_l(t)} \tag{2.8}$$

3 1 DOF DC motor model with load

Figure 3.1a shows the mechanical part of a DC motor, the gearbox and the load and Figure 3.1b its frontal view.

Let the load be attached to the motor axis by a reduction gearbox of $r \in (0,1]$ reduction factor and $\eta \in (0,1]$ efficiency. Then, the angular velocity i$\dot{\theta}_L(t)$ and the load torque $\tau_L(t)$ at the gearbox output will be related respectively to $\dot{\theta}_m(t)$ and $\tau_l(t)$ in the motor axis (or gearbox input) by the following equations:

$$\dot{\theta}_L(t) = r\dot{\theta}_m(t) \tag{3.1a}$$

$$\tau_L(t) = \frac{\eta}{r}\tau_l(t) \tag{3.1b}$$

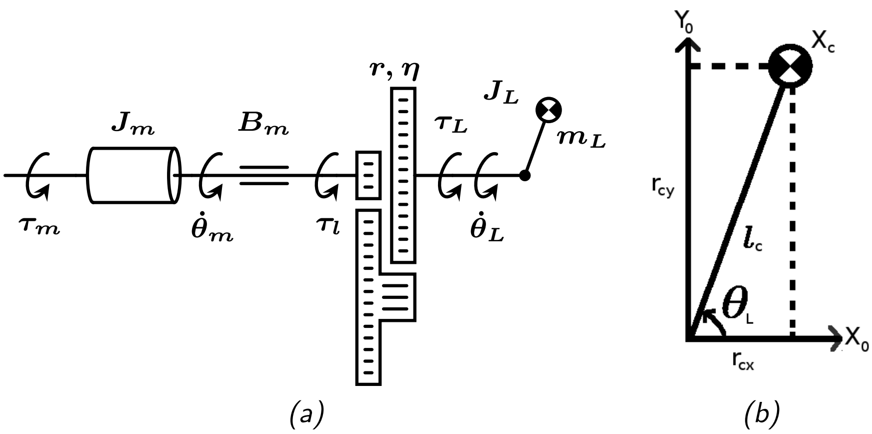

Figure 3.1: (a) Side and (b) frontal view of the axis, gearbox and DC motor load.

The kinetic (K) and potential (V) energy of a rigid body that rotates around axis z of the inertial reference system $\mathcal{S}_0$ are

$$K = \frac{1}{2}mv_c^T v_c + \frac{1}{2}\widetilde{w}^T I_c \widetilde{w} \tag{3.2a}$$

$$V = mgr_{cy} \tag{3.2b}$$

where m is the mass, v_c the linear velocity of the center of mass, $\widetilde{w} = R_z w$, with w the angular velocity of the object and R_z the rotation matrix between the $\mathcal{S}_0$ and $\mathcal{S}_c$ reference systems, r_{cy} the y coordinate of the center of mass position, g the earth's gravitational constant and I_c the constant inertia tensor around the center of mass

$$I_c = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{3.3}$$

As it is a 1 DOF mechanism, the Cartesian coordinates of the center of mass refereed to the system $\mathcal{S}_0$ are,

$$X_c = l_c \begin{bmatrix} \cos\theta_L(t) \\ \sin\theta_L(t) \\ 0 \end{bmatrix} \tag{3.4}$$

where l_c is the distance between the origin of the inertial reference system $\mathcal{S}_0$ and the center of mass.

By deriving with respect to the time, the linear velocity of the center of mass v_c is obtained. From the derivate of the elemental rotation matrix around axis z, the angular velocity $\widetilde{w}$ is obtained. Therefore,

$$v_c = \dot{X}_c = l_c \dot{\theta}_L(t) \begin{bmatrix} -\sin\theta_L(t) \\ \cos\theta_L(t) \\ 0 \end{bmatrix} \tag{3.5a}$$

$$\widetilde{w} = \dot{\theta}_L(t)\vec{k} = \dot{\theta}_L(t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.5b}$$

$$r_{cy} = l_c \sin\theta_L(t) \tag{3.5c}$$

$$\widetilde{w}^T I_c \widetilde{w} = I_{zz}\dot{\theta}_L^2(t) \tag{3.5d}$$

$$v_c^T v_c = l_c^2 \dot{\theta}_L^2(t) \tag{3.5e}$$

where I_{zz} is the moment of inertia around axis z of the reference system fixed at the center of mass $\mathcal{S}_c$.

The dynamic equation of the load can be obtained by different ways. One of them is the Euler-Lagrange formulation, that will be explained as follows.

The Lagrangian L is defined as $L = K - V$, where K is the kinetic energy and V the potential energy of the mechanism. Therefore, it can be demonstrated that the dynamic equation of every mechanism of a rigid body can be obtained by the Euler-Lagrange equation. For a 1 DOF mechanism:

$$\tau_L(t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_L(t)}\right) - \frac{\partial L}{\partial \theta_L(t)} \tag{3.6}$$

where $\tau_L(t)$ is named generalized force or torque, in our case the load torque.

Therefore, by taking into account the aforementioned developments,

$$\tau_L(t) = J_L \ddot{\theta}_L(t) + mgl_c \cos\theta_L(t) \tag{3.7}$$

where $J_L = I_{zz} + ml_c^2$.

By expressing $\theta_L(t)$ and $\tau_L(t)$ in terms of $\theta_m(t)$ and $\tau_l(t)$, the equation of the object movement given by Equation 3.7 can be rewritten as

$$\tau_l(t) = \frac{r^2 J_L}{\eta}\ddot{\theta}_m(t) + \frac{r}{\eta}\tau_g(t) \tag{3.8}$$

where $\tau_g(t) = mgl_c \cos(r\theta_m(t))$ is the gravitational torque compensation.

By combining Equation 3.8 in the simplified motor equation given by Equation 2.8

$$\boxed{\frac{k_m}{R_m}u_m(t) = J_{eff}\ddot{\theta}_m(t) + B\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(t)} \tag{3.9}$$

where J_{eff} is the effective moment of inertia

$$\boxed{\begin{aligned} J_{eff} &= J_m + \frac{r^2 J_L}{\eta} && (3.10a)\\ B &= B_m + \frac{k_b k_m}{R_m} && (3.10b)\\ \tau_g(t) &= mgl_c \cos\left(r\theta_m(t)\right) && (3.10c) \end{aligned}}$$

4 Introduccing an external force to the 1 DOF DC motor model with load

If an external force f_e and a moment of forces n_e, is applied to the end (P_e) of 1 DOF robot, then the torque that appears at the joint is

$$\boxed{\tau_e(t) = J^T(\theta_L(t))F_e(t)} \tag{4.1}$$

where J represents the Jacobian of the robot and F_e the external forces torsor,

$$J(\theta_L(t)) = \begin{bmatrix} -l\sin\left(\theta_L(t)+\alpha_c\right) \\ l\cos\left(\theta_L(t)+\alpha_c\right) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.2a}$$

$$F_e(t) = \begin{bmatrix} f_{ex}(t) \\ f_{ey}(t) \\ f_{ez}(t) \\ n_{ex}(t) \\ n_{ey}(t) \\ n_{ez}(t) \end{bmatrix} \tag{4.2b}$$

where l is the distance between the origin of the inertial reference system $\mathcal{S}_0$ and the point where the force is applied, and α_c is a constant angle between the P_e and r_c vectors.

The demonstration of Equation 4.1 is obtained by the virtual works of D'Alembert. This demonstration is not going to be done in this manuscript.

In any case,

$$\tau_e(t) = l\left(-f_{ex}(t)\sin\left(\theta_L(t)+\alpha_c\right) + f_{ey}(t)\cos\left(\theta_L(t)+\alpha_c\right)\right) + n_{ez}(t) \tag{4.3}$$

The force $f_e(t)$ can be decomposed in a tangential force and a normal force to the robot's movement. The normal force does not perform any work, so the external force will only influence the tangential force.

If $f_{eT}(t)$ is the module of the tangential force, then the effective force will be

$$\begin{aligned} f_{ex}(t) &= \pm f_{eT}(t)\sin\left(\theta_L(t)+\alpha_c\right) && (4.4a)\\ f_{ey}(t) &= \mp f_{eT}(t)\cos\left(\theta_L(t)+\alpha_c\right) && (4.4b) \end{aligned}$$

then

$$\tau_e(t) = \mp l f_{eT}(t) + n_{ez}(t) \tag{4.5}$$

By introducing Equation 4.5 in the simplified equation of the DC motor with load given by Equation 3.9

$$\boxed{\frac{k_m}{R_m}u_m(t) = J_{eff}\ddot{\theta}_m(t) + B\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(t) - \frac{r}{\eta}\left(l f'_{eT}(t) - n_{ez}(t)\right)} \tag{4.6}$$

where $f'_{eT}(t) = \pm f_{eT}(t)$.

In what follows $f_{eT}(t)$ will be used instead of $f'_{eT}(t)$ to indicate the module of the tangential force with sign, understanding that the sign will be positive when the tangential force is applied in counter-clock wise in the xy plane of the inertial reference system $\mathcal{S}_0$.

5 Direct torque control

The direct torque control technique consists of defining a structure with two feedback loops as shown in Figure 5.1. The internal loop function is to linearize the system and the external one to control the linear system to satisfy some design specifications. Although in this Section a 1 DOF system is studied, it is easy to generalize it to a multivariable control system for a n DOFs robot.

The dynamic equation of the motor with load obtained in Section 3 can be rewritten as

$$u(t) = J_{eff}\ddot{\theta}_m(t) + B\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(\theta_m(t)) \tag{5.1}$$

where $u(t) = \dfrac{k_m}{R_m}u_m(t)$ and $u_m(t)$ is the voltage applied to the DC motor.

The direct torque control technique consists of applying the following non-linear feedback control loop,

$$\boxed{u(t) = J_{eff}v(t) + B\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(\theta_m(t))} \tag{5.2}$$

where $v(t)$ represents an input signal that must be designed.

By substituting Equation 5.2 in Equation 5.1,

$$v(t) = \ddot{\theta}_m(t) \tag{5.3}$$

where it was taken into account that $J_{eff} \neq 0$.

This means that the system will behave as a double integrator, so $v(t)$ can be designed by linear control design techniques. However, for this to happen, a perfect cancellation of the inertial terms, viscous and gravitational friction, must occur. Unfortunately, in the real world, errors and uncertainty in the modelling are typical, so the general form of the cancelation can be expressed as

$$v(t) = \ddot{\theta}_m(t) - \sigma(t) \tag{5.4}$$

where $\sigma(t)$ is a time-invariant linear function and, in general, dependent of $\theta_m(t)$ and $\dot{\theta}_m(t)$.

There are several control techniques that take into account that $\sigma(t) \neq 0$, as the robust and adaptive control techniques. An easy technique that allows to apply linear control techniques is to consider $\sigma(t)$ as a perturbation and to design controllers that solve the problem of suppressing perturbations. In this document none of these techniques are explained, but it must be understood that any function that is not taken into account in the model, as the static friction, can be treated as a perturbation.

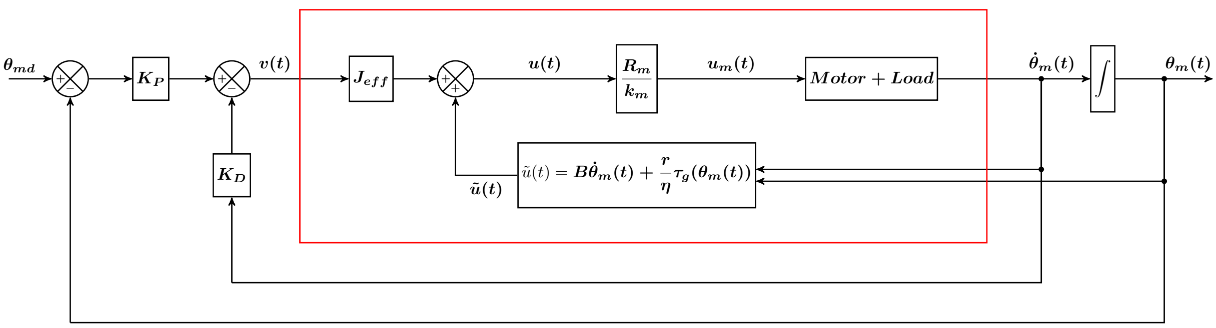

Figure 5.1: Block diagram of a direct torque control of position.

The design of the control signal $v(t)$ depends on the application to be implemented.

If a position control, where the desired position θ_{md} is known, wants to be implemented by assuming that the cancellation is perfect, that means $\sigma(t) = 0$, a type P-D controller can be designed as the one shown in Figure 5.1. In this example, $v(t)$ is

$$v(t) = K_P(\theta_{md} - \theta_m(t)) - K_D\dot{\theta}_m(t) \tag{5.5}$$

where K_P are K_D design constants.

In Figure 5.1 the linearized system is shown in red. It ideally satisfies Equation 5.3, $v(t) = \ddot{\theta}_m(t)$.

6 Gravity torque compensation

This section focuses on designing a controller to compensate the gravity torque. The design will be based on the direct torque control technique shown in Section 5. However, instead of controlling the position it will be focused on controlling the velocity.

Let the initial conditions of angular position and velocity of a motor be $\theta_m(t_0^-)$ and $\dot{\theta}_m(t_0^-)$ respectively. An external force $F_e(t)$ is applied to the end of the robot during a time interval $[t_0, t_f]$, and no force is applied when $t > t_f^-$.

During the interval at which the external force $F_e(t)$ is applied, the load moves to $\theta_m(t_f^-)$ and its velocity is $\dot{\theta}_m(t_f^-)$. The objective of the gravity torque control is to design a controller that satisfies the condition that the robot is stopped at any position when there is no external force. Therefore, the condition to be met is that the velocity must be zero when there is no external force:

$$\dot{\theta}_{md} = 0, \quad t > t_f^- \tag{6.1}$$

where $\dot{\theta}_{md}$ represents the desired angular velocity.

Let us consider the dynamic equation of the DC motor with load and an external force applied to its end studied in Section 4 given by Equation 4.6 applied to the $t \in [t_0, t_f]$ interval, and the dynamic equation of the motor with load when there is not external force applied, that is, when $f_{eT} = n_{ez} = 0$, then

$$u(t) = \begin{cases} J_{eff}\ddot{\theta}_m(t) + B\dot{\theta}_m(t) + \dfrac{r}{\eta}\tau_g(\theta_m(t)) - \dfrac{r}{\eta}\left(l f_{eT}(t) - n_{ez}(t)\right) & t \in [t_0, t_f] \quad (6.2a) \\ J_{eff}\ddot{\theta}_m(t) + B\dot{\theta}_m(t) + \dfrac{r}{\eta}\tau_g(\theta_m(t)) & t > t_f^- \quad (6.2b) \end{cases}$$

where $u(t) = \dfrac{k_m}{R_m} u_m(t)$, $u_m(t)$ is the input voltage of the motor and $\tau_g(\theta_m(t)) = mgl_c \cos\left(r\theta_m(t)\right)$ represents the gravity torque.

The direct torque control law studied in Section 5 can be expressed as

$$u(t) = J_{eff} v(t) + B\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(\theta_m(t)) \tag{6.3}$$

where $v(t)$ represents an input signal that must be designed.

As shown in Section 5, by substituting Equation 6.3 in Equation 6.2b, the system will behave as a double integrator, that is, a linearization will be produced and will satisfy the dynamic equation $v(t) = \ddot{\theta}_m(t)$.

With the objective of using the viscous damping as a break, which makes velocity close to zero and therefore the $B\dot{\theta}_m(t)$ term almost negligible, the control law can be modified as:

$$u(t) = J_{eff} v(t) + \frac{r}{\eta}\tau_g(\theta_m(t)) \tag{6.4}$$

Therefore, by assuming a perfect cancellation of the gravity torque $\tau_g(\theta_m(t))$, the system will be linearized for $t > t_f^-$, with the following equation

$$v(t) = \ddot{\theta}_m(t) + \frac{B}{J_{eff}}\dot{\theta}_m(t) \tag{6.5}$$

With the objective of stopping the robot as fast as possible, a linear velocity controller can be designed, for example, a derivative feedback controller in the parallel loop,

$$v(t) = -\frac{K_D}{J_{eff}}\dot{\theta}_m(t) \tag{6.6}$$

where K_D is a design constant.

In summary, the control law $u(t)$ will be

$$\boxed{u(t) = -K_D\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(\theta_m(t))} \tag{6.7}$$

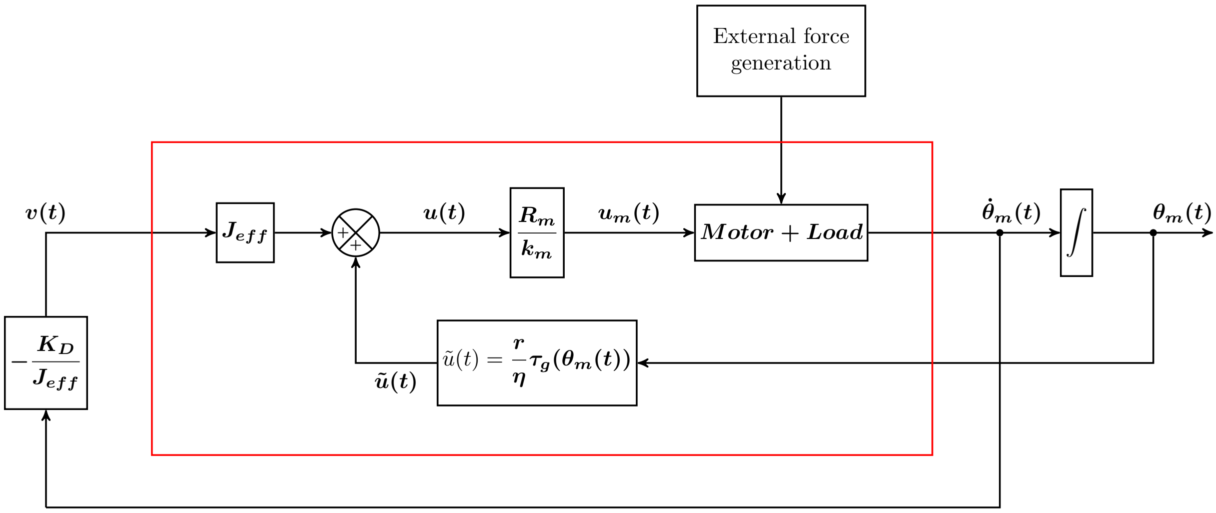

Figure 6.1: Block diagram of the gravity torque compensation based on the direct torque control technique.

Figure 6.1 shows the block diagram of the control system. To show the similarity with the direct torque control technique studied in Section 5, the linearized part that satisfies Equation 6.5 is shown in red.

7 Introduction to haptic feedback

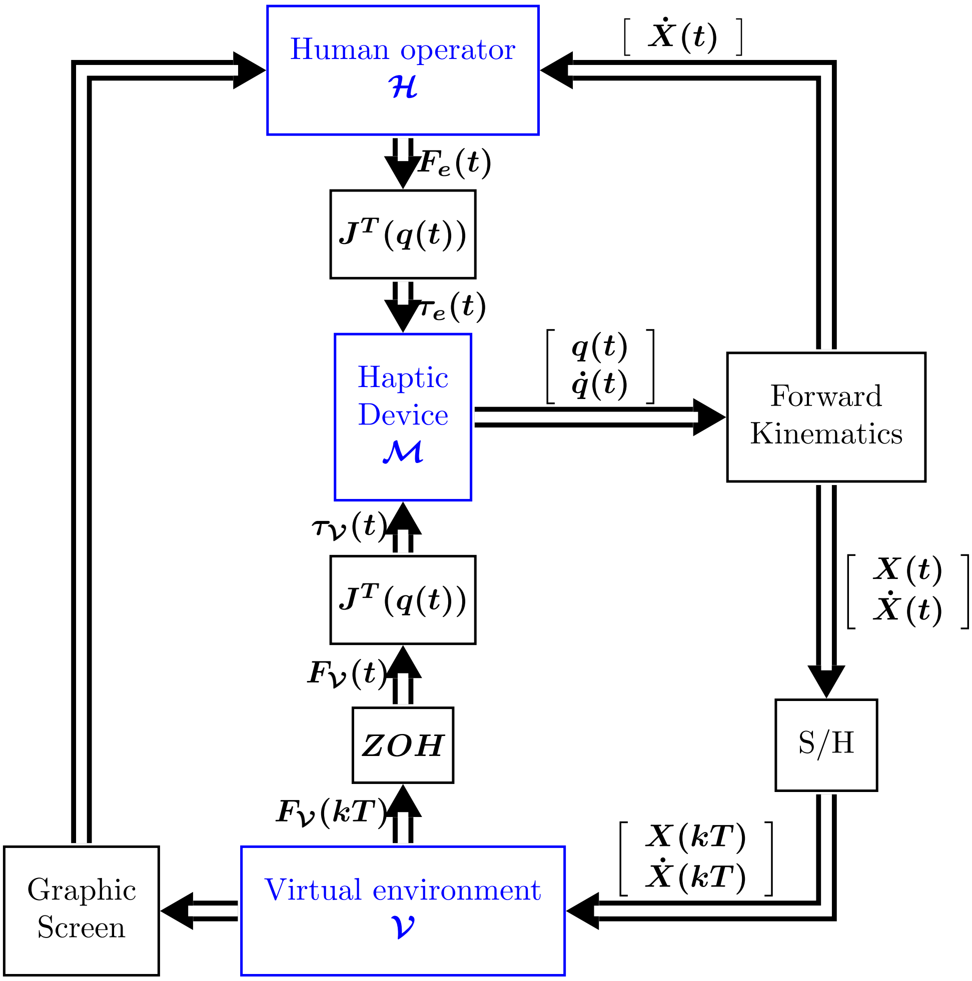

Figure 7.1: Block diagram of a haptic feedback.

Figure 7.1 shows a simplified block diagram of a system with haptic feedback. There is a master robot $\mathcal{M}$ that is actuated at its end by a human operator $\mathcal{H}$. The human operator exerts an external force $F_e(t)$ to the end of the robot. On the other hand, there is a real or virtual environment that opposes the movement of the master. In this section a virtual environment $\mathcal{V}$ is assumed. In this environment there are objects that have some mechanical characteristics, as rigidity, elasticity, mass, texture, viscous friction, etc.

The **haptic feedback** consists of allowing the human operator to perceive tactile sensations related to the mechanical characteristics of the virtual environment. In this case the master robot is named **haptic device**. In Figure 7.1, $J^T(q(t))$ represents the transpose of the robot Jacobian, which depends on the generalized coordinates $q(t)$ and ZOH represents a necessary Zero Order Hold device because the haptic device is a continuous system while the virtual environment is discrete. Figure 7.2 shows an example of input and output signals of a ZOH dependent of time.

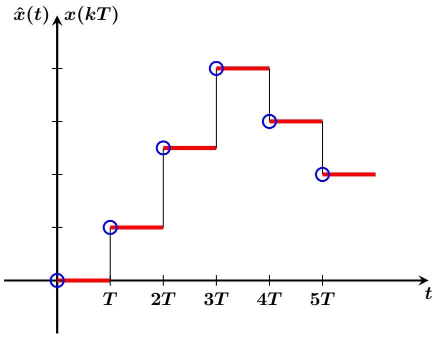

Figure 7.2: ZOH device: $x(kT)$ is the input (blue) and $\hat{x}(t)$ is the output (red).

In Figure 7.1, a visual feedback has been included, although, in general, it could be any other perceptive signal or no signal at all.

There will be haptic sensations when there is a relationship between the exerted force over an object and the movement of the object, that is, between the force and velocity. The simple contact with an object is not considered haptic sensation, although in some cases this is an ideal limit of haptic sensation. An example is the case of pushing or pressing a rigid wall, because in this case a static reaction force is produced, that is, there is no movement. In reality, walls are very rigid, but when pushing with the hand, they produce a movement because of the hand elasticity. If the robot that pushes the wall is rigid, an elastic wall could be emulated. In this case, a movement related to the force applied to the virtual wall will be produced, therefore it should be considered as a haptic sensation as there is a movement involved.

Probably the simplest example is the case in which the human operator of the haptic device pushes a virtual object in a rough surface of a virtual environment. If the object is a rigid mass, when a force $f_{\mathcal{V}}$ is applied , it produces a movement given by

$$f_{\mathcal{V}}(t) = m_{\mathcal{V}}\ddot{X}(t) + B_{\mathcal{V}}\dot{X}(t) \tag{7.1}$$

where $X(t)$ represents the displacement, $B_{\mathcal{V}}$ the viscous damping constant of the surface and $m_{\mathcal{V}}$ the object mass.

The haptic sensation allows the human operator that is displacing a virtual object to perceive a reaction force of the moving object, $-f_{\mathcal{V}}(t)$.

To study the movement of the virtual object, it is necessary to establish a relationship between the Cartesian coordinates of the end of the haptic device and the coordinates of the virtual object. Therefore, there must exist a relationship between the inertial reference system of the haptic device $\mathcal{S}_{\mathcal{M}}$ and the inertial reference system of the virtual environment $\mathcal{S}_{\mathcal{V}}$. The simplest solution to this problem is to consider that a specific position Q_0 of the end of the haptic device, in coordinates of the system $\mathcal{S}_{\mathcal{M}}$, corresponds to the origin of coordinates of the virtual reference system $\mathcal{S}_{\mathcal{V}}$, and that the relationship between $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}_{\mathcal{V}}$ consists of a constant rigid movement, that is, that it could be expressed as a constant translation and rotation.

In this case, let d_0^{n0} be the point Q_0 expressed in coordinates relative to $\mathcal{S}_{\mathcal{M}}$, that is, $\overrightarrow{OQ_0} = d_0^{n0}$, and R_0^{n0} to the rotational matrix of $\mathcal{S}_{\mathcal{V}}$ with respect to $\mathcal{S}_{\mathcal{M}}$.

Therefore, the point Q of the end of the haptic device, whose coordinates in the $\mathcal{S}_{\mathcal{M}}$ system are d_0^n, is transformed in the coordinates related to the $\mathcal{S}_{\mathcal{V}}$ system as $d_0^n = R_0^{n0} d_{n_0}^n + d_0^{n0}$ where $d_{n_0}^n = \overrightarrow{Q_0 Q}$ is expressed in coordinates of the $\mathcal{S}_{\mathcal{V}}$ system. Therefore, in the $\mathcal{S}_{\mathcal{V}}$ system,

$$d_{n_0}^n = \left(R_0^{n0}\right)^T \left(d_0^n - d_0^{n0}\right) \tag{7.2}$$

The point Q of the end of the haptic device can be in contact with a virtual object in a specific point by exerting a force over it. This force could displace the object according to Equation 7.1. The $X(t)$ displacement should be expressed in coordinates of the $\mathcal{S}_{\mathcal{V}}$ system. If the virtual object is initially at X_0, then $X(t) = X_{n_0}(t) - X_0$, where $X_{n_0}(t) = d_{n_0}^n(t)$ and $X_0 = d_{n_0}^n(t_0)$, being t_0 the time instant at which the end of the haptic device contacts the first time the virtual object.

Therefore, it is obvious that the **collision detection problem** should be solved prior to the **displacement problem**. In any case, the **collision problem** should be solved during all the interval in which the object is pushed. If at any time the end of the haptic device stops being in touch with the object, the applied force should be zero.

The relationship between the end of the haptic device and the virtual object can be very different. For example, a translational displacement of point Q of the haptic device could correspond to a rotation displacement of the virtual object. Or a translational displacement of point Q could correspond to roll a disc in the virtual environment, interpreting that the force applied by the haptic device is a torque applied to the disc, as if the haptic device will be pressing the gas pedal of a car. This means that, although virtual objects have their analogy in the real world, their movements do not need to be identical to the haptic device. Nevertheless, the haptic feedback must be such that the human operator of the haptic device perceives the virtual world with the maximum fidelity. This requirement is called **haptic transparency problem**. Nonetheless, it is very difficult to completely eliminate the haptic distortion that exists between the perception and what it should be perceived because of the no linearities of actuators and sensors resolution.

The kinematic relationship given by Equation 7.2 is only valid to the aforementioned conditions. It could be useful when the forward kinematics problem of the haptic device must be solved to obtain the virtual environment coordinates. But in general, the relationship between $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}_{\mathcal{V}}$ could be any other.

In any case, there will always be errors related to the exact position and velocity of the end of the haptic device and virtual environment because of numerical errors of solving the forward kinematic problem and the movement discretization. In fact, the example of pushing a mass with viscous friction, which dynamic equation is given by Equation 7.1, is continuous, while the one simulated in the virtual environment is discrete. All these issues pose a problem related to the haptic feedback named **haptic stability problem**. The haptic instability is typical of a low sampling rate, and can be perceived as oscillations in the end of the haptic device. It is important to note that the haptic perception requires a frequency of 1 kHz.

8 Virtual model of a torsion spring with viscous damping

In this Section, the objects programed in the virtual environment, $\mathcal{V}$, will establish a relationship between the virtual torque $\tau_{\mathcal{V}}(kT)$ and the virtual angular displacement $\theta_{\mathcal{V}}(kT)$ by means of a difference equation obtained as a discretization of a real object. At the end of the Section, a relationship between the virtual angular displacement $\theta_{\mathcal{V}}(kT)$ and the real angular displacement $\theta_m(kT)$ of the haptic device is shown.

The virtual object will consist of an ideal torsion spring with viscous damping. In this case, the analogous real object will satisfy

$$\tau_{\mathcal{V}}(t) = k_{\mathcal{V}}\theta_{\mathcal{V}}(t) + B_{\mathcal{V}}\dot{\theta}_{\mathcal{V}}(t) \tag{8.1}$$

with the initial condition $\theta_{\mathcal{V}}(t_0^-)$ and t_0 the initial time, $k_{\mathcal{V}}$ is the torsion or elasticity constant and $B_{\mathcal{V}}$ the viscous damping constant. Function $\tau_{\mathcal{V}}(t)$ represents the torque that must be applied to the real object to be at time t in the angular position $\theta_{\mathcal{V}}(t)$ with angular velocity $\dot{\theta}_{\mathcal{V}}(t)$.

Because the virtual spring is part of the haptic system, it will be supposed that the initial time t_0 corresponds to the time instant in which the collision is produced. It will also be assumed that $\tau_{\mathcal{V}}(t) = 0$ for $t \le t_0^-$, that is $\dot{\theta}_{\mathcal{V}}(t_0^-) = 0$ and $\theta_{\mathcal{V}}(t_0^-) = 0$. However, it should be allowed that the applied torque in the instant of the collision could not be zero, that is, $\tau_{\mathcal{V}}(t_0^+) \neq 0$.

The mathematical model of the virtual object can be obtained by discretizing the differential equation given by Equation 8.1. Any discretization will always be an approximation of the continuous version.

The solution of the differential equation given by Equation 8.1 is,

$$\theta_{\mathcal{V}}(t) = \theta_{\mathcal{V}}(t_0^-)e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}(t-t_0)} + \frac{1}{B_{\mathcal{V}}}e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}t}\int_{t_0}^{t} e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}\nu}\,\tau_{\mathcal{V}}(\nu)d\nu \tag{8.2}$$

The solution given by Equation 8.2 is met for every t, so it will also be true for $t = kT$ and

$t = (k+1)T$,

$$\theta_{\mathcal{V}}(k) = \theta_{\mathcal{V}}(t_0^-)e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}(kT - t_0)} + \frac{1}{B_{\mathcal{V}}}e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}kT}\int_{t_0}^{kT} e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}\nu}\tau_{\mathcal{V}}(\nu)d\nu \tag{8.3a}$$

$$\theta_{\mathcal{V}}(k+1) = \theta_{\mathcal{V}}(t_0^-)e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}((k+1)T - t_0)} + \frac{1}{B_{\mathcal{V}}}e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}(k+1)T}\int_{t_0}^{(k+1)T} e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}\nu}\tau_{\mathcal{V}}(\nu)d\nu \tag{8.3b}$$

where, for simplicity on the notation, $\theta_{\mathcal{V}}(kT) = \theta_{\mathcal{V}}(k)$ and $\theta_{\mathcal{V}}((k+1)T) = \theta_{\mathcal{V}}(k+1)$.

The temporal index k does not take values $k = 0, 1, 2, \cdots$ because what is interesting is to take into account the collision related to the real time t, that is, to impose that the initial time $t_0 \neq 0$. To easy the study it will be assumed that $t_0 = k_0 T$, that is, that the collision is produced at a multiple integer of the sampling period T. Therefore,

$$k \in \{k_0, k_0+1, k_0+1, \cdots\} \tag{8.4}$$

Multiplying Equation 8.3a by $e^{-\frac{k_{\mathcal{V}}T}{B_{\mathcal{V}}}}$, subtracting Equation 8.3b and taking into account that $\int_{t_0}^{(k+1)T} = \int_{t_0}^{kT} + \int_{kT}^{(k+1)T}$, the following difference equation is obtained:

$$\theta_{\mathcal{V}}(k+1) = e^{-\frac{k_{\mathcal{V}}T}{B_{\mathcal{V}}}}\theta_{\mathcal{V}}(k) + \frac{1}{B_{\mathcal{V}}}e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}(k+1)T}\int_{kT}^{(k+1)T} e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}\nu}\tau_{\mathcal{V}}(\nu)d\nu \tag{8.5}$$

This difference equation is an exact discretization of the continuous solution. If $\tau_{\mathcal{V}}(t)$ is constant during the time interval $[kT, (k+1)T)$ with value $\tau_{\mathcal{V}}(kT)$, Equation 8.5 can be rewritten as:

$$\theta_{\mathcal{V}}(k+1) = e^{-\frac{k_{\mathcal{V}}T}{B_{\mathcal{V}}}}\theta_{\mathcal{V}}(k) + \frac{1}{B_{\mathcal{V}}}e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}(k+1)T}\tau_{\mathcal{V}}(k)\int_{kT}^{(k+1)T} e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}\nu}d\nu \tag{8.6}$$

where, because of commodity on the notation, $\tau_{\mathcal{V}}(kT) = \tau_{\mathcal{V}}(k)$.

Solving the integral,

$$\theta_{\mathcal{V}}(k+1) = e^{-\frac{k_{\mathcal{V}}T}{B_{\mathcal{V}}}}\theta_{\mathcal{V}}(k) + \frac{1}{k_{\mathcal{V}}}e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}(k+1)T}\tau_{\mathcal{V}}(k)\left(e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}(k+1)T} - e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}kT}\right) \tag{8.7}$$

Simplifying,

$$\theta_{\mathcal{V}}(k+1) = e^{-\frac{k_{\mathcal{V}}T}{B_{\mathcal{V}}}}\theta_{\mathcal{V}}(k) + \frac{1}{k_{\mathcal{V}}}\left(1 - e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T}\right)\tau_{\mathcal{V}}(k) \tag{8.8}$$

Therefore, the discretization of the torsion spring model with damping viscous can be rewritten as:

$$a_1(T)\theta_{\mathcal{V}}(k+1) + a_2(T)\theta_{\mathcal{V}}(k) = \tau_{\mathcal{V}}(k) \tag{8.9}$$

where the sampling period dependent constants, $a_1(T)$ and $a_2(T)$ are

$$a_1(T) = \frac{k_{\mathcal{V}}}{1 - e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T}} \tag{8.10a}$$

$$a_2(T) = -\frac{k_{\mathcal{V}}e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T}}{1 - e^{-\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T}} \tag{8.10b}$$

Equation 8.9 represents a real system, and therefore a causal system, where the input is $\tau_{\mathcal{V}}(k)$ and the output $\theta_{\mathcal{V}}(k)$. However, the virtual objects must be built in such a way that the input is $\theta_{\mathcal{V}}(k)$ and the output is $\tau_{\mathcal{V}}(k)$, or more concretely the reaction force of the virtual object, that is, $-\tau_{\mathcal{V}}(k)$. This means that they cannot be completely analogous to real systems. In fact, to obtain $\tau_{\mathcal{V}}(k)$ it is necessary to know $\theta_{\mathcal{V}}(k+1)$, that is, a future value, and therefore unknown at $t = kT$.

A solution to this problem of no causality of the virtual objects is to provide a delay of one sampling period in the signal$\tau_{\mathcal{V}}(k)$. This can be acceptable if T is low enough. In this case, the spring model could be

$$\boxed{\tau_{\mathcal{V}}(k) = a_1(T)\theta_{\mathcal{V}}(k) + a_2(T)\theta_{\mathcal{V}}(k-1)} \tag{8.11}$$

with $k = k_0+1, k_0+2, \cdots$, the initial condition $\theta_{\mathcal{V}}(k_0) = 0$ and where the sampling period dependent constants, $a_1(T)$ y $a_2(T)$, are given by

$$a_1(T) = \frac{k_{\mathcal{V}} e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T}}{e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T} - 1} \tag{8.12a}$$

$$a_2(T) = -\frac{k_{\mathcal{V}}}{e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T} - 1} \tag{8.12b}$$

It is important to notice that $\theta_{\mathcal{V}}(k)$ is related with the real displacement of the end of the haptic device. If the haptic device is a robot of 1 DOF, it is mandatory to measure $\theta_m(t)$, that is, the angular position of the motor axis. If $\theta_{\mathcal{V}}(t) = \theta_m(t)$, the no causality problem is also present, although it could be avoided with an angular speed sensor, that is, by measuring $\dot{\theta}_m(t)$. If this is the case, the virtual model could be obtained from Equation 8.1 as

$$\boxed{\tau_{\mathcal{V}}(k) = k_{\mathcal{V}}\theta_{\mathcal{V}}(k) + B_{\mathcal{V}}\dot{\theta}_{\mathcal{V}}(k)} \tag{8.13}$$

where $\theta_{\mathcal{V}}(k) = \theta_m(k)$ and $\dot{\theta}_{\mathcal{V}}(k) = \dot{\theta}_m(k)$.

Typically there are not velocity sensors (tachometers) installed in the haptic devices, and a discrete overdue approximation of the derivative is used. The most simple approximation is the one of Euler:

$$\dot{\theta}_m(t) \approx \frac{\theta_m(k) - \theta_m(k-1)}{T} \tag{8.14}$$

By substituting this approximation in Equation 8.1, assuming $\theta_{\mathcal{V}}(t) = \theta_m(t)$ and grouping terms:

$$\boxed{\tau_{\mathcal{V}}(k) = b_1(T)\theta_{\mathcal{V}}(k) + b_2(T)\theta_{\mathcal{V}}(k-1)} \tag{8.15}$$

with the initial condition, where

$$b_1(T) = k_{\mathcal{V}} + \frac{B_{\mathcal{V}}}{T} \tag{8.16a}$$

$$b_2(T) = -\frac{B_{\mathcal{V}}}{T} \tag{8.16b}$$

For this model of the spring, the virtual models of Equation 8.11 and Equation 8.15 are exclusively different on the constants. We can check that constants $b_i(T)$ are an approximation of the exact ones $a_i(T)$ for low sampling periods. It can be seen by taking into account the definition of the exponential, by substituting the first order linear approximation in $a_i(T)$ as

$$e^{\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T} = 1 + \frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T + \frac{1}{2}\left(\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T\right)^2 + \frac{1}{6}\left(\frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T\right)^3 + \cdots \approx 1 + \frac{k_{\mathcal{V}}}{B_{\mathcal{V}}}T \tag{8.17}$$

Finally, it is important to mention that the relationships $\theta_{\mathcal{V}}(t) = \theta_m(t)$ and $\dot{\theta}_{\mathcal{V}}(t) = \dot{\theta}_m(t)$ do not need to be met. These relationships depend on the problem to be solved, and therefore it is a problem of designing the virtual environment. In the case of a 1 DOF haptic system with a motor and a reduction gearbox in the joint, it seems convenient to assume that $\theta_{\mathcal{V}}(t)$ meets the angular position at the gearbox output, and that the stable position of the spring is not zero, but a constant value θ_{L0}. Therefore, the relationship between $\theta_{\mathcal{V}}(t)$ and $\theta_m(t)$, and between $\dot{\theta}_{\mathcal{V}}(t)$ and $\dot{\theta}_m(t)$ are represented as

$$\boxed{\begin{aligned} \theta_{\mathcal{V}}(t) &= r\theta_m(t) - \theta_{L0} && (8.18a)\\ \dot{\theta}_{\mathcal{V}}(t) &= r\dot{\theta}_m(t) && (8.18b)\end{aligned}}$$

where $r \in [0, 1]$ is the reduction ration and $\theta_{L0} = r\theta_m(t_0)$.

By including the relationship between the virtual object and the real world given by Equation 8.18 in the virtual object Equation 8.11, a difference equation that defines the virtual object programmed in the virtual environment is obtained,

$$\boxed{\tau_{\mathcal{V}}(k) = ra(T)\theta_m(k) + r(k_{\mathcal{V}} - a(T))\theta_m(k-1) - ra(T)\theta_{m0}} \tag{8.19}$$

where $\theta_{m0} = \theta_m(k_0)$, $a(T) = a_1(T)$ and it was taken into account that $a_1(T) + a_2(T) = k_{\mathcal{V}}$.

On the other hand, exact discrete solutions of a specific real model are not approximations obtained from the Euler approximation of the derivate. In any case, the important objective are not the mathematics developed in this Section, but to solve the **transparency problem**, in the sense that the haptic perception of the human operator matches with the virtual model implemented.

9 Haptic system with a 1 DOF haptic device and a virtual torsion spring with viscous damping

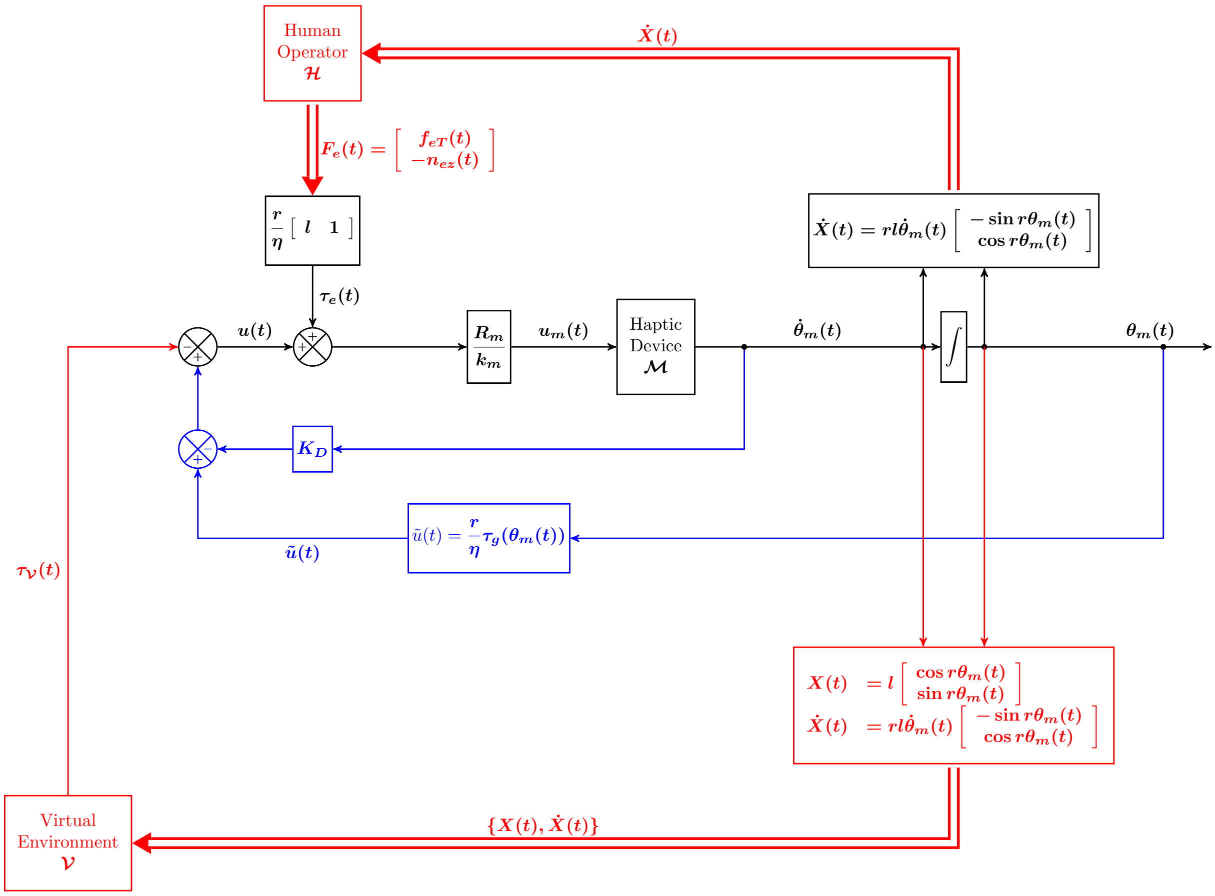

Figure 9.1: 1 DOF Haptic system with gravity compensation

In Section 6 a gravity compensation controller was designed. However, when there is a haptic feedback, the external force applied by the human operator depends on the linear velocity of the end of the haptic device, as shown in Figure 7.1.

Figure 9.1 shows the block diagram of a haptic system including the controller studied in Section 6 for a haptic device of 1 DOF. This Figure assumes that the virtual environment implements objects that depend on the Cartesian coordinates $\{X(t), \dot{X}(t)\}$, and that a sampler is located at is input and a ZOH at its output.

When a haptic feedback is included, the external torque $\tau_e(t)$, that appears at the joint of the haptic device when the human operator applies force $F_e(t)$, depends on the joint coordinates, that is

$$\tau_e(t) = \tau_e(\theta_m(t), \dot{\theta}_m(t)) \tag{9.1}$$

because $\tau_e = J^T F_e$, where J represents the robot's Jacobian, which depends on the angular position. F_e depends on the linear velocity of the end of the haptic device that depends on the angular position and the angular velocity of the motor axis.

The equation which represents the haptic system is the same that the one obtained in Section 4, Equation 4.6, of the motor with load when an external force is applied, with the sole difference that $\tau_e(t)$ depends on the position and angular velocity.

$$u(t) = J_{eff}\ddot{\theta}_m(t) + B\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(\theta_m(t)) - \frac{r}{\eta}\tau_e(t) \tag{9.2}$$

where $u(t) = \dfrac{k_m}{R_m}u_m(t)$, being $u_m(t)$ the input voltage to the motor.

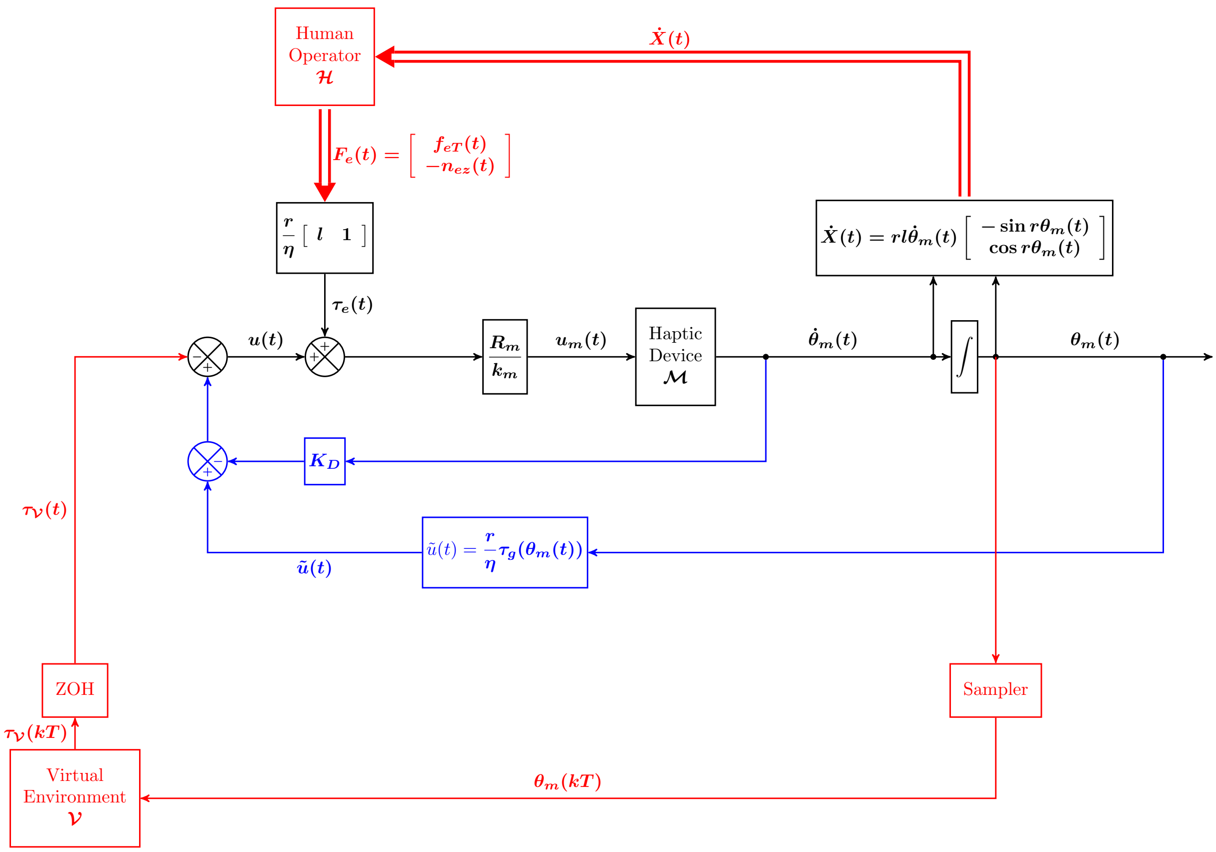

Figure 9.2: 1 DOF Haptic system with gravity compensation

In what follows, it will be studied how to incorporate the virtual environment on the dynamics of the haptic system. Figure 9.2 shows the block diagram of a 1 DOF haptic device where the need for solving the forward kinematics problem has been eliminated, because the virtual object to study will be represented in joint coordinates.

If the virtual environment consists of a torsion spring with friction, as the one studied in Section 8, Equation 8.11 or Equation 8.15 can be studied with a ZOH device as

$$\tau_{\mathcal{V}}(k) = a_1(T)\theta_{\mathcal{V}}(k) + a_2(T)\theta_{\mathcal{V}}(k-1) \tag{9.3}$$

being $t \in [kT, (k+1)T)$ with $k \in \{k_0+1, k_0+2, \cdots\}$ and $t_0 = k_0 T$ the time instant of the collision.

If the relationship between $\theta_{\mathcal{V}}(t)$ and $\theta_m(t)$ are such that the virtual object follows Equation 8.19, when the ZOH is included, the virtual object equation in $t \in [kT, (k+1)T)$ with $k \in \{k_0+1, k_0+2, \cdots\}$, can be rewritten as:

$$\tau_{\mathcal{V}}(k) = a_{\mathcal{V}}\theta_m(k) + b_{\mathcal{V}}\theta_m(k-1) - a_{\mathcal{V}}\theta_{m0} \tag{9.4}$$

where $a_{\mathcal{V}} = ra(T)$ and $b_{\mathcal{V}} = r(k_{\mathcal{V}} - a(T))$. It is also met that $a_{\mathcal{V}} + b_{\mathcal{V}} = rk_{\mathcal{V}}$.

To include this discrete virtual model in the continuous equation of the haptic system given by Equation 9.2 is problematic. There are different techniques for the study of hybrid systems. One of them is to assume that the discrete part consist of an exact discretization of the continuous model, such that it is assumed that the discrete part will not conduct to important errors. This assumes that the virtual object is continuous. In our case, it means to use the model of Equation 8.1 that together with Equation 8.18 conduct to

$$\tau_{\mathcal{V}}(t) = rk_{\mathcal{V}}\theta_m(t) + rB_{\mathcal{V}}\dot{\theta}_m(t) - rk_{\mathcal{V}}\theta_{m0} \tag{9.5}$$

If the control law given by Equation 6.7 is considered,

$$\boxed{u(t) = -K_D\dot{\theta}_m(t) + \frac{r}{\eta}\tau_g(\theta_m(t)) - \tau_{\mathcal{V}}(t)} \tag{9.6}$$

By substituting the control law given by Equation 9.6 in the dynamic equation of the haptic system given by Equation 9.2, the differential equation that represents the haptic system is given by

$$\boxed{J_{eff}\ddot{\theta}_m(t) + (B + K_D + rB_{\mathcal{V}})\dot{\theta}_m(t) + rk_{\mathcal{V}}\theta_m(t) - \frac{r}{\eta}\tau_e(\theta_m(t), \dot{\theta}_m(t)) = rk_{\mathcal{V}}\theta_{m0}} \tag{9.7}$$

The solution to this differential equation depends on the knowledge of the effect of the human operator at the output of the gearbox, that is, $\tau_e(\theta_m(t), \dot{\theta}_m(t))$.

References

[1] R. M. Murray, Z. Li, and S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. CRC Press, Inc., 1994.